MATH11007 SHEET 15: PARAMETRIC CURVES, ARCLENGTH ETC.

Set on Tuesday, February 14: Qs 2, 4, and 5.

(1) For the following curves, find (i) y' and (ii) y''.
   (a) $x(t) = 1 + t$, $y(t) = t + t^2$.
   (b) $x(t) = 1 + 1/t^2$, $y(t) = 3t + 1$.
   (c) $x(t) = a\cos(2t)$, $y(t) = b\sin t$.
   (d) $x(\theta) = a\cos^2\theta$, $y(\theta) = b\sin^3\theta$.
   (e) $x(t) = \mathrm{e}^{-t}\cos t$, $y(t) = \mathrm{e}^{-t}\sin t$.

   Plot these curves using `Maple`'s `plot[parametric]` command. Then, for (a)-(d), find an equation of the form $y = f(x)$ for the curve. Hence verify your answers.

(2) Find the cartesian coordinates of the highest point of the curve of parametric equation $x(t) = \mathrm{e}^t$, $y(t) = t - t^2$.

(3) Find the equations of the tangent line and of the normal line to the following curves at the specified point.
   (a) $x(t) = a\mathrm{e}^{-t}$, $y(t) = b\mathrm{e}^{2t}$ at $t = 0$.
   (b) $x(\theta) = a\cos^4\theta$, $y(\theta) = a\sin^4\theta$ at $\theta = \pi/3$.

(4) Consider the curve defined by $x(t) = t^2 - 1$ and $y(t) = t^3 - t$. Locate the points where the tangent line is (i) horizontal and (ii) vertical. Show that, at the point where the curve crosses itself, the two tangent lines are mutually orthogonal.

(5) Find the length of the following curves.
   (a) $x(t) = \mathrm{e}^t\cos t$, $y(t) = \mathrm{e}^t\sin t$, $z(t) = \mathrm{e}^t$, $0 \le t \le 3$.
   (b) $x(t) = \ln\sqrt{1+t^2}$, $y(t) = \arctan t$, $z(t) = 1$, $0 \le t \le 1/\sqrt{3}$.
   (c) $x(\theta) = 2\cos\theta + \cos(2\theta) + 1$, $y(\theta) = 2\sin\theta + \sin(2\theta)$, $0 \le \theta \le \pi/4$.

(6) The position of a particle at time t is given by

$$x(t) = \frac{t^2}{2}\,, \quad y(t) = \frac{1}{9}\,(6t+8)^{\frac{3}{2}}\,, \quad z(t) = t\,, \quad 0 \le t \le 4\,.$$

   Find the distance travelled by the particle.

(7) Show that the curvature of a straight line is 0.

(8) Find the points of maximum curvature of the curve $x(t) = 2\tan t$, $y(t) = \tan^2 t$.

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition* Chapters 37 and 38, Mc-Graw–Hill, 1999.