MATH11007 SHEET 16: INTRODUCTION TO SEVERAL VARIABLES

Set on Tuesday, February 21: Qs 1, 4 and 6

(1) For the following functions, (i) determine the maximal domain, (ii) the range and (iii) sketch the level curves.
  (a) $f(x,y) = \sqrt{x+y}\,.$
  (b)
$$f(x,y) = \sqrt{\frac{2y}{x^2+y^2-1}}\,.$$

(2) Use `Maple`'s `plot3d` command to plot the graphs of the following functions.
  (a) $f(x,y) = y^2$.
  (b) $f(x,y) = x - y$.
  (c)
$$f(x,y) = \frac{1}{x^2+y^2+1}\,.$$

(3) Use `Maple`'s `plot3d` command with the option `style=contour` to plot the level curves of the following functions.
  (a) $f(x,y) = |x| - |y|$.
  (b) $f(x,y) = y^2 - x^2(x+1)$.

(4) Let $f(x,y) = x^2/(x^2+y^2)$. Show that
$$\lim_{(x,y)\to(0,0)} f(x,y)$$
does not exist.

(5) Use the definition of partial derivative as a limit to find $f_x(x,y)$ and $f_y(x,y)$ if
  (a) $f(x,y) = x^2 - xy + 2y^2$.
  (b) $f(x,y) = \sqrt{3x-y}$.

(6) Find the first partial derivatives of the following functions:
  (a) $f(x,y) = y\ln x$ for $x > 0$.
  (b) $p(s,t) = \cos(st)$.
  (c) $r(p,q) = p^q$ for $p > 0$.
  (d) $x(a,b) = (a+b)/(a-b)$ for $a \neq b$.
  (e) $t(x,y,z) = xy/\sqrt{x^2+y^2+z^2}$ where x, y, z are not all zero.

(7) Evaluate the second partial derivatives of the following functions. In each case, you should find that the mixed second partial derivatives are equal.
  (a) $f(x,y) = y\ln x$.

(b) $g(u, v, w) = \sqrt{u + v^2 + w^3}$.

(8) Use implicit differentiation to find z_x and z_y if $yz = \ln(x + z)$.

(9) Suppose that

$$v(x, y) = \int_a^x u(t, y)\,\mathrm{d}t + k(y)$$

where a is some real number and k is a function of the variable y. Find $v_x(x, y)$ and $v_{xx}(x, y)$.

(10) The Laplace equation for a function u of two variables x and y is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

This equation arises in many physical contexts, e.g. to model the distribution of electric charges in equilibrium, or the shape of an elastic membrane at rest. It is an equation involving partial derivatives, and is therefore called a *partial differential equation*. Such equations are typically harder to solve than ordinary differential equations but, given a function, it is easy to test whether or not it satisfies the equation. Which of the following functions are solutions of Laplace's equation within their maximal domains?

(a) $\sin x \sin y$; (b) $\sinh x \sin y$; (c) $\ln(x^2 + y^2)$; (d) $x^2 + y^2$.

(11) Evaluate the second partial derivatives of $u(x, y, z) = x^{yz}$.

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition* Chapters 48 and 52, Mc-Graw–Hill, 1999.