MATH11007 SHEET 17: THE CHAIN RULE AND THE GRADIENT.

Set on Tuesday, February 28: Qs 1, 2 and 4

(1) Find the gradient of each of the following functions

(a) $f(x,y) = \mathrm{e}^{xy}$; (b) $h(s,t) = \arctan(s/t)$; (c) $x(a,b,c) = \ln[(a^2+b^2)/c^2]$.

(2) In each of the following, find $\mathrm{d}w/\mathrm{d}t$ in two ways: (i) using the chain rule and (ii) substituting to get w as a function of t, then differentiating.
 (a) $w = x^2y^2$ where $x = \sin t$, $y = \cos t$.
 (b) $w = p/q$ where $p = \mathrm{e}^t$, $q = t^2$.
 (c) $w = xy/z$ where $x = t\cos t$, $y = t\sin t$, $z = t$.

(3) In each of the following cases, find $\partial z/\partial u$ and $\partial z/\partial v$ in two ways: (i) using the chain rule and (ii) substituting to get z as a function of u, v and then differentiating.
 (a) $z = \sqrt{x^2+y^2}$, $x = \mathrm{e}^{uv}$, $y = 1 + u^2\cos v$.
 (b) $z = \arctan(x/y)$, $x = 2u + v$, $y = 3u - v$.

(4) Atmospheric temperature decreases exponentially with height. The temperature in a certain region can be approximated by the function $T(x,y,z) = \mathrm{e}^{-z}/(5+x^2+y^2)$, where x, y are horizontal coordinates and z is height. A balloon is moving through the region, and its position at time t is given by $x = t$, $y = 2t$, $z = t - t^4$. Use the chain rule to find how fast the temperature at the position of the balloon at time t is changing.

(5) Suppose that

$$z = \frac{x-y}{x+y}\,,$$

where $x = uvw$ and $y = u^2 + v^2 + w^2$. Use the chain rule to find z_u, z_v and z_w.

(6) A *partial differential equation* is an equation for some unknown function that relates the function's partial derivatives. The equation

$$\text{(PDE)}\qquad \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0\,, \quad x \in \mathbb{R}\,, \quad t > 0\,,$$

where c is a constant, is one example.
 (a) Show that $u(x,t) = \sin(x - ct)$ is a solution of (PDE).
 (b) Set $c = 1$. Using `Maple` or otherwise, display the graphs of $u(x,0)$, $u(x,1)$ and $u(x,2)$ on the same (x,u) plot. You should find that the graphs are the same, except for a shift.

(c) To find the general solution of (PDE), introduce the new variables

$$p = x - ct \quad \text{and} \quad q = x + ct\,,$$

and set

$$u(x,t) = U(p,q)\,.$$

Show that, for every (p,q),

$$\frac{\partial U}{\partial q}(p,q) = 0$$

and deduce the general solution of (PDE).

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition* Chapters 48 and 52, Mc-Graw–Hill, 1999.