MATH11007 SHEET 19: LOCAL EXTREMA IN SEVERAL VARIABLES

Set on Tuesday, March 13: Qs 1, 3 and 4.

(1) In Newtonian mechanics, a force field $\mathbf{f}$ is said to be *conservative* if it can be expressed in the form

$$\mathbf{f}(\mathbf{x}) = -\nabla V(\mathbf{x})$$

for some function V depending only on position $\mathbf{x}$. The function V is called the *potential*. Consider the following potentials:
  (a) $V(x,y,z) = 2x^2 - 5xy + 3y^2 + 6x - 7y + z^2$;
  (b) $V(x,y,z) = x^2 + 4y^2 + z^2 - 4xy - 4yz + 2xz - 4x + 8y - 4z$;
  (c) $V(x,y,z) = x^2 + y^2 + z^2 - 8x + 16y - 4z$;
  (d) $V(x,y,z) = (5x^2 + 2y^2)\exp(-x^2 - y^2 + z^2)$.

In each case, (i) find the corresponding force and (ii) determine whether there are any equilibrium points, i.e. points where the force vanishes.

(2) Use MAPLE's `plot3d` function to plot the graph of

$$V(x,y,z) = \cos x \cosh y + \frac{z^2}{2}$$

as a function of x and y in the range $-10 \leq x \leq 10$, $-3 \leq y \leq 3$, when $z = 0$. Find the gradient ∇V. Does this function have a minimum?

(3) For the following functions, find all the critical points and classify them.
  (a) $f(x,y) = x^2 - y^2 + 4y$.
  (b) $f(x,y) = x^2 - 3xy + 2y^2 + 4y$.
  (c) $u(s,t) = s^2t^2 + s^2 - t^2$.
  (d) $\phi(\xi,\eta) = \xi^2\eta^2 - \xi^2 - \eta^2$.

(4) When it rains on the surface of cartesian equation $z = 1/x + 1/y + xy$ (for x, $y > 0$), a puddle will form in just one place. Where?

(5) The graph of the function $f(x,y) = 1/xy$ for x, $y > 0$ is a surface in 3-dimensional space. Find the point of this surface which is closest to the origin.

(6) Give an example of a function $f(x,y)$ such that the discriminant

$$\mathscr{D}(f) := f_{xx}f_{yy} - f_{xy}^2$$

vanishes at a critical point that is (a) a maximum, (b) a minimum, (c) neither.

(7) Let $\mathbf{x}(t)$ denote the position at time t of a particle whose velocity is proportional to the direction of greatest decrease of the potential V. In other words,

$$\dot{\mathbf{x}} = -k\nabla V\left(\mathbf{x}(t)\right)$$

for every $t > 0$, where k is a constant.

Let the potential be as in Question 2. Show that, when $z(0) = 0$, the trajectory lies in the xy plane and that the curve $y = y(x)$ describing the particle's path satisfies the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\cot x \tanh y\,.$$

Find the solution of this equation, and hence plot the trajectory passing through the point

$$\mathbf{x} = \begin{pmatrix} \frac{\pi}{2} \\ \operatorname{arctanh}(1/2) \\ 0 \end{pmatrix}\,.$$

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition* Chapters 48 and 52, Mc-Graw–Hill, 1999.