MATH11007 SHEET 21: POLAR COORDINATES

Set on Monday, April 23: Qs 2,3 and 4.

(1) Using `Maple`'s `plots[polarplot]` command or otherwise, sketch the following curves expressed in polar coordinates.
  (a) The cardioid: $r = 1 + \sin\theta$.
  (b) The limaçon: $r = 1 + 2\cos\theta$.
  (c) The rose with three petals: $r = \cos(3\theta)$.
  (d) The lemniscate: $r^2 = \cos(2\theta)$.

(2) The following curves are expressed in polar coordinates. In each case, find a formula for the curvature as a function of θ.

(a) $r = \mathrm{e}^{\theta}$; (b) $r = \sin(\theta)$; (c) $r^2 = 4\cos(2\theta)$; (d) $r = 3\sin\theta + 4\cos\theta$.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
  (a) $f(x, y) = xy$.
  (b) $f(x, y) = \sin(x + y)$.
  (c) $f(x, y) = \sqrt{1 + x^2 + y^2}$.
  (d)

$$f(x,y) = \begin{cases} \ln(x^2 + y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases} .$$

In each case, let $u(r, \theta) = f(r\cos\theta,\, r\sin\theta)$, and compute the partial derivatives u_r and u_θ.

(4) Let $f = f(x, y)$ be a function defined on some two-dimensional domain. Let $u(r, \theta) = f(r\cos\theta,\, r\sin\theta)$. Show that

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\Big|_{x = r\cos\theta, y = r\sin\theta} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} .$$

(5) Sketch the region

$$R = \left\{(x, y) \in \mathbb{R}^2 \,:\, x \geq 0,\, 1 \leq x^2 + y^2 \leq 2\right\}$$

and, by changing to polar coordinates, compute

$$\iint_R x^2 \mathrm{d}x\mathrm{d}y \,.$$

(6) Sketch the region

$$R = \left\{(x,y) \in \mathbb{R}^2 \ : \ y \geq x,\ 1 \leq x^2 + y^2 \leq 2\right\}$$

and, by changing to polar coordinates, compute

$$\iint_R \frac{xy}{x^2+y^2} \mathrm{d}x\mathrm{d}y \,.$$

(7) Let $a > 0$. Evaluate the following integrals

(a)

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \mathrm{d}y\mathrm{d}x \,.$$

(b)

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y \,.$$

(c)

$$\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} x\,\mathrm{d}x\mathrm{d}y \,.$$

(8) Find the area of the region enclosed by the curve $r^2 = \cos\theta$. Then compute the volume under the surface of equation $z = \sqrt{1-x^2-y^2}$ above the region.

(9) A cylindrical hole of radius 1 is bored through the center of a sphere of radius 2. What volume is removed?

(10) Let $n \in \mathbb{N}$, $f(x,y) = 1/r^n$, where $r = \sqrt{x^2+y^2}$, and

$$R(a,b) = \left\{(x,y) \in \mathbb{R}^2 \ : \ a \leq \sqrt{x^2+y^2} \leq b\right\}$$

where $0 < a < b$. By changing to polar coordinates, compute

$$I(a,b) := \iint_{R(a,b)} f(x,y)\,\mathrm{d}x\mathrm{d}y \,.$$

For which values of n does the limit

$$\lim_{a \to 0} I(a,b)$$

exist?

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition* Chapter 41, Mc-Graw–Hill, 1999.
2. Serge Lang, *Calculus of Several Variables, Second Edition*, Chapter VII, §3, Addison–Wesley, Reading, 1979.