MATH11007 SHEET 22: TRIPLE INTEGRALS, SPHERICAL COORDINATES

Set on Monday, April 30: Qs 1, 2 and 4.

(1) Compute the volume of the region defined by the following inequalities:

$$0 \leq x \leq \sqrt{1-y^2-z^2}\,, \quad 0 \leq y \leq \sqrt{1-z^2}\,, \quad 0 \leq z \leq 1\,.$$

(2) Compute the integral

$$\int_0^{\pi} \int_0^{\sin\theta} \int_0^{\rho\cos\theta} \rho^2 \, \mathrm{d}z \, \mathrm{d}\rho \, \mathrm{d}\theta\,.$$

(3) Let T be the tetrahedron defined by the inequalities

$$0 \leq x \leq 1-y-z\,, \quad 0 \leq y \leq 1-z\,, \quad 0 \leq z \leq 1\,.$$

Find

$$\iiint_T \mathrm{e}^{x+y+z} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z\,.$$

(4) Compute the volume inside the cone

$$\sqrt{x^2+y^2} \leq z \leq 1$$

by using spherical coordinates.

(5) Let $0 < a < 1$.

(a) Compute the mass of a spherical ball of radius a if the density at any point is equal to a constant k times the distance of that point to the center.

(b) Compute the integral of the function

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

over the spherical shell of inside radius a and outside radius 1.

(6) Let n be a positive integer, and let $f(x,y,z) = 1/\rho^n$, where

$$\rho = \sqrt{x^2+y^2+z^2}\,.$$

(a) Compute the integral of the function

$$f(x,y,z) = 1/\rho^n$$

over the region contained between two spheres of radii a and b respectively, with $0 < a < b$.

(b) For which value of n does this integral approach a limit as $a \to 0$?

(7) Find the mass and the center of mass of the cylinder

$$C := \left\{(x,y,z)\,:\,0 \le z \le 1\,,\,0 \le x^2 + y^2 \le 1\right\}\,,$$

assuming its density is uniform.

(8) Find the mass and the center of mass of a circular plate of radius a, assuming its density is proportional to the square of the distance from the center.

(9) Find the center of mass of a (filled) cone of height h, whose base has a radius equal to a, assuming its density is proportional to the distance from the base.

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition* Chapter 57, Mc-Graw–Hill, 1999.
2. Serge Lang, *Calculus of Several Variables, Second Edition*, Chapter VII, §3, Addison–Wesley, Reading, 1979.