# MATH11007 NOTES 1: BEFORE CALCULUS

Abstract. Some techniques and concepts that you need in order to study Calculus.

## 1. What is the unit about?

1.1. **Alernative names.** The unit might have been called

"Differential and Integral Calculus"

or

"Infinitesimal Calculus"

The concept of "infinity" is at the very heart of our subject.

1.2. **Why study Calculus?** The techniques and concepts that you will learn will enable you to tackle in a very elegant way some computations that would otherwise be enormously difficult.

**Example 1.1.** *Compute*

$$s_n := \sum_{k=1}^{n} \frac{1}{(2k-1)^2}$$

*for* $n = 1, 2, 4$ *etc.*

*As* $n$ *gets larger, you need to do more work to compute the sum. Surprisingly, there is a wonderful trick for computing the sum when* $n = \infty$*. Then one finds*

$$\lim_{n\to\infty} s_n = \frac{\pi^2}{8} \,.$$

One might say with some simplification that many computational tasks become easier in the limit as some number becomes infinitely large or small. Infinity becomes an intellectual device for making complicated problems simpler. In the foregoing example, the calculation of the infinite sum is an application of the theory of Fourier series and reduces to the evaluation of some simple integrals.

1.3. **Historical development.** Calculus was invented more or less simultaneously and independently by Newton (1665–1667) and Leibniz (1672–1674). It seems fair to say that it depended for its development on Descartes' earlier idea of linking geometry and algebra through the cartesian plane. Newton was very secretive about the new Calculus, whereas Leibniz published his ideas widely, using notation and concepts very close to those we still use today. The new Calculus progressed rapidly on the continent, thanks to the contributions made by the Bernoulli family and Euler, among others. Euler wrote the first Calculus textbook *Introductio in* *analysin infinitorum* (1748). The material that we shall cover in this unit is not substantially different.

1.4. **Pitfalls of infinity!** One has to take great care when working with infinity.

**Example 1.2.** *Here is a "proof" that* $0 = 1$*:*

$$0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$
$$= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots$$
$$= 1 + 0 + 0 + 0 + \cdots = 1\,.$$

Infinite sums cannot be treated like finite sums. Historically, apparent paradoxes arising from Calculus led to the development of Analysis in the 19th Century. The purpose of Analysis is to provide a firm, rigourous foundation for Calculus.

1.5. **The Fundamental Theorem of Calculus.** This is the cornerstone of the unit, on which everything else rests. It states that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)\,.$$

The statement contains all the usual ingredients: numbers ($a$), variables ($x$ and $t$), functions ($f$), derivatives and integrals. To make sense of the statement, we need to examine the meaning of all these ingredients:

- What is a derivative? A limit.
- What is an integral? A limit.
- What is a limit? A number.
- What is a number?
- What is a function?

1.6. **From whole numbers to real numbers.** We take as our starting point the *natural numbers*

$$\mathbb{N} := \{0,\ 1,\ 2,\ 3,\ \ldots\}\ .$$

If we want to solve equations of the form

$$x + n = 0\,, \quad n \in \mathbb{N},$$

we are led very quickly to the larger set of the *whole numbers* (german *Zahlen*)

$$\mathbb{Z} := \{\ldots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}\ .$$

Then we might like to solve

$$qx - p = 0\,, \quad p, q \in \mathbb{Z},\ .$$

This leads to the *rational numbers* (quotients)

$$\mathbb{Q} := \{p/q\ :\ p \in \mathbb{Z},\ 0 \neq q \in \mathbb{N}\}\ .$$

Finally (?), we might like to solve

$$x^2 = n \in \mathbb{N}\,.$$

If so, then we must look to a set of numbers larger than $\mathbb{Q}$. This larger set is the set of the *real numbers*; it is denoted by $\mathbb{R}$.

## 2. The real line; intervals and inequalities (see [1], Chapter 1)

What are the real numbers? The question will be examined in some depth in the Analysis unit. Here, we take a pragmatic approach and learn to manipulate them.

2.1. **The order relations (inequalities).** $\mathbb{R}$ is an ordered set, i.e. there is a relation "$<$" such that, for any two real numbers $a$ and $b$, exactly one of the following holds:

$$a < b \qquad \text{“}a \text{ is less than } b\text{”}$$
$$a > b \qquad \text{“}a \text{ is greater than } b\text{”}$$
$$a = b \qquad \text{“}a \text{ equals } b\text{”}$$

We use the abbreviated form $a \leq b$ for "$a$ is less than or equal to $b$". When handling the order relation, the following rules must be adhered to (here $a$, $b$ and $c$ are all real numbers):

(1) If $a < b$ then $a + c < b + c$.
(2) If $a \leq b$ and $c \geq 0$ then $ac \leq bc$.
(3) If $a \leq b$ and $c < 0$ then $ac \geq bc$.

2.2. **The absolute value of a real number.**

*Notation* . The symbol

$$:=$$

is a shorthand for "by definition".

The *absolute value* of the real number $a$ is defined by

$$|a| := \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} .$$

One may think of $|a|$ as the size of the number $a$. We have the *triangle inequality* for real numbers, i.e. for every $a$ and $b$,

$$|a + b| \leq |a| + |b| .$$

We emphasise that the triangle inequality is a theorem— not an axiom. It can be proved from the rules above by considering separately the cases (1) $a, b \geq 0$, (2) $a \geq 0$, $b < 0$, (3) $a, b < 0$.

2.3. **Intervals.** Sets of real numbers of the following form will play a prominent part in the unit. They are called *intervals*.

*Notation* . For $a < b$,

$$[a, b] := \{x \in \mathbb{R} \,:\, a \leq x \text{ and } x \leq b\} .$$
$$(a, b] := \{x \in \mathbb{R} \,:\, a < x \text{ and } x \leq b\} .$$
$$[a, b) := \{x \in \mathbb{R} \,:\, a \leq x \text{ and } x < b\} .$$
$$(a, b) := \{x \in \mathbb{R} \,:\, a < x \text{ and } x < b\} .$$

**Example 2.1.** *Express the set*

$$A := \{a \in \mathbb{R} \,:\, |a| \leq 1\}$$

*as an interval.*

Answer*:* $A = [-1, 1]$.

**Example 2.2.** *"Solve" the inequality*

$$\frac{3x-1}{x} < 4\,.$$

*In other words, express the set*

$$A := \left\{ x \in \mathbb{R} \,:\, \frac{3x-1}{x} < 4 \right\}$$

*as a union of intervals.*

Solution: *Using the rules, we can write*

$$\frac{3x-1}{x} < 4 \iff 3 - \frac{1}{x} < 4 \iff -\frac{1}{x} < 1 \iff \frac{1}{x} > -1\,.$$

*There are then two possibilities: either $x > 0$ or $x < 0$. From the foregoing,*

1. *Case $x > 0$: Then $x \in A$.*
2. *Case $x < 0$: Then $x \in A$ if $x < -1$.*

*So*

$$A = (0, \infty) \cup (-\infty, -1)\,.$$

**Example 2.3.** *Solve the following inequality*

$$|3x + 4| < |2x - 1|\,. \tag{2.1}$$

Solution: *It would be awkward to work with the absolute value, so we begin by noting that the inequality is equivalent to*

$$|3x+4| < \begin{cases} 2x - 1 & \text{if } x \geq 1/2 \\ 1 - 2x & \text{otherwise} \end{cases}\,.$$

*So we need to treat the cases $x < 1/2$ and $x \geq 1/2$ separately.*

*Suppose first that $x \geq 1/2$. Then the inequality is equivalent to*

$$1 - 2x < 3x + 4 < 2x - 1\,.$$

*That is:*

$$1 - 2x < 3x + 4 \;\text{ and }\; 3x + 4 < 2x - 1\,.$$

*The first inequality is satisfied by $x > -3/5$ and the second by $x < -5$. No number can satisfy* both *inequalities simultaneously. Hence there can be no solution of (2.1) such that $x \geq 1/2$.*

*Suppose then that $x < 1/2$. Inequality (2.1) is then equivalent to*

$$2x - 1 < 3x + 4 < 1 - 2x\,.$$

*Hence*

$$-5 < x < -3/5\,.$$

*The set of numbers satisfying (2.1) is therefore the interval $(-5, -3/5)$.*

## 3. Functions

For our purposes, by a *function*, we mean a collection of three ingredients, say $A$, $B$ and $f$.

(1) $A$ is a set of numbers called the *domain* of the function,
(2) $B$ is a set of numbers called the *codomain* of the function,
(3) $f$ is a "mapping" between $A$ and $B$ such that, to every $x \in A$, the mapping assigns a unique element $y = f(x) \in B$.

We shall sometimes write

$$f \,:\, A \to B$$

to indicate a function. The domain and the codomain are part of the definition of a function, but it is often the case that the mapping $f$ by itself is referred to as the "function". Hence, one often reads statements like: "Find the derivative of the function $f(x) = x$ with respect to $x$" etc. Such abuses are acceptable as long as the context makes them unambiguous.

Given a mapping $f$, the *maximal domain* of $f$ is the largest set of real numbers $A$ that can serve as the domain of a function with mapping $f$ and codomain $\mathbb{R}$.

**Example 3.1.** *$A = B = \mathbb{R}$ and*

$$f(x) = |x| \,.$$

*The maximal domain of $f$ is $\mathbb{R}$.*

**Example 3.2.** *The* Heaviside function *$H$ is defined on the domain $\mathbb{R}$ by*

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \end{cases} .$$

*Note that neither*

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 < x \end{cases}$$

*nor*

$$H(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } 0 \le x \end{cases}$$

*would define functions on the domain $\mathbb{R}$.*

Some functions are particularly important in the applications. Let us mention

- *Polynomials*: These are functions which can be written in the form

  $$f(x) = a_0 + a_1 x + \cdots + a_d x^d$$

  for some numbers $a_j$, $0 \le j \le d$, $d \in \mathbb{N}$. If $a_d \neq 0$, we say that $f$ is a polynomial of *degree* $d$ with *leading coefficient* $a_d$. The maximal domain of any polynomial is $\mathbb{R}$. We say that $r$ is a *root* or a *zero* of the polynomial $f$ if $f(r) = 0$.
- *Rational functions:* These are functions of the form

  $$f(x) = \frac{p(x)}{q(x)}$$

  where $p$ and $q$ are polynomials. The maximal domain of $f$ is then

  $$A := \mathbb{R} \backslash \{x \in \mathbb{R} \,:\, q(x) = 0\} \,.$$

  We exclude the roots of the denominator— dividing by zero is a crime.

- *Trigonometric functions:* These are functions defined geometrically by means of the familiar circle, e.g.

$$\sin x\,,\ \cos x\,,\ \tan x,\ \cot x\,,\ \text{etc.}$$

You will be expected to know the most elementary properties of these functions. See the example below.
- More tricky are the functions defined by inverting the above maps for suitable choices of the domain and the codomain. As an example, take the function

$$f(x) = x^2$$

with domain $\mathbb{R}$ and codomain $\mathbb{R}_+ := [0, \infty)$. For $y \geq 0$, the quadratic equation

$$y = f(x)$$

admits exactly one non-negative root, and so we can define the *square root function* as the map that assigns $x$ to $y$. We write this as

$$x = \sqrt{y}\,.$$

The maximal domain of the square root function is then $\mathbb{R}_+$. The inverse trigonometric functions

$$\arcsin x\,,\ \ \arccos x\,,\ \ \arctan x\,,\ \ \text{etc.}$$

are defined similarly.

**Example 3.3.** *Evaluate* $\csc(7\pi/6)$ *"by hand".*

Solution: *Looking at the trigonometric circle, we see that*

$$\sin(7\pi/6) = -\sin(\pi/6)\,.$$

*Hence*

$$\csc(7\pi/6) = -2\,.$$

## References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.
2. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.