MATH11007 NOTES 2: LIMITS

Abstract. What limits are, and how to use them.

You will be used to statements like

$$\lim_{x\to 0}\frac{1-x}{1+x}=1 \quad\text{and}\quad \lim_{x\to\infty}\frac{1}{x}=0\,.$$

What do they mean exactly?

1. Definitions

Let x_0, $L\in\mathbb{R}$, $f:A\to\mathbb{R}$. We assume that x_0 is either in the set A, or else is an *accumulation* point of the set A, i.e.

$$\forall\,\delta>0\,,\ \exists\, x\in A\ \text{ such that }\ |x-x_0|<\delta\,.$$

Definition 1.1. We say that L is the *limit* of $f(x)$ as x tends to x_0 if

$$\forall\,\varepsilon>0\,,\ \exists\,\delta>0\ \text{ such that }\ x\in A \text{ and } |x-x_0|<\delta \implies |f(x)-L|<\varepsilon\,.$$

Notation . We write interchangeably

$$\lim_{x\to x_0} f(x)=L\,,$$

$$f(x)\xrightarrow[x\to x_0]{} L$$

or

$$f(x)\to L \quad \text{as } x\to x_0\,.$$

This definition can be extended to the following cases:

(1) $x_0\in\mathbb{R}$, $L=\pm\infty$.
(2) $x_0=\pm\infty$, $L\in\mathbb{R}$.
(3) $x_0=\pm\infty$, $L=\pm\infty$.

For example,

$$\lim_{x\to x_0} f(x)=\infty$$

means: For every $\varepsilon>0$, there exists $\delta>0$ such that

$$x\in A\ \text{ and }\ |x-x_0|<\delta \implies f(x)>\frac{1}{\varepsilon}\,.$$

More

Notation .

$$\lim_{x \to x_0+} f(x) = L$$

means: the limit of $f(x)$ as x approaches x_0 from the right (i.e. $x > x_0$) is L.

$$\lim_{x \to x_0-} f(x) = L$$

means: the limit of $f(x)$ as x approaches x_0 from the left (i.e. $x < x_0$) is L.

2. Examples

Example 2.1. *Show that*

$$\lim_{x \to 1} (x+1) = 2\,.$$

Solution: *Here $x_0 = 1$, $L = 2$ and $f(x) = x + 1$ with, say, $A = \mathbb{R}$. According to the definition, we have to show that, for every $\varepsilon > 0$, no matter how small, we can find a number $\delta > 0$ such that $|x - 1| < \delta$ implies $|f(x) - 2| < \varepsilon$. So, let ε be* any *positive number. Let $x \in \mathbb{R}$ be such that $|x - 1| < \varepsilon$. Then*

$$|f(x) - 2| = |x + 1 - 2| = |x - 1| < \varepsilon\,.$$

So by taking $\delta = \varepsilon$, we have found a positive number δ for which the required implication does hold.

Example 2.2. *Show that*

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0\,.$$

Solution: *Here $x_0 = \infty$, $L = 0$ and $f(x) = (\sin x)/x$ with, say, $A = (0, \infty)$. According to the definition, we have to show that, for every $\varepsilon > 0$, no matter how small, we can find a number $\delta > 0$ such that $x > \delta$ implies $|f(x)| < \varepsilon$. So, let ε be* any *positive number. Let $x > 1/\varepsilon$. Then*

$$|f(x)| = \frac{|\sin x|}{|x|} \le \frac{1}{|x|} < \varepsilon\,.$$

So by taking $\delta = 1/\varepsilon$, we have found a positive number δ for which the required implication does hold.

3. A theorem about limits

It would be painful to have to compute every limit from the definition. The following simple result enables the computation of many limits from known results without the need for ε and δ.

Suppose that

$$\lim_{x \to x_0} f(x) = L_f \quad \text{and} \quad \lim_{x \to x_0} g(x) = L_g\,.$$

Then

$$\lim_{x \to x_0} [f(x) + g(x)] = L_f + L_g\,, \tag{3.1}$$

$$\lim_{x \to x_0} f(x)g(x) = L_f L_g \tag{3.2}$$

and, if g does not vanish in its domain and $L_g \neq 0$,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_f}{L_g}\,. \tag{3.3}$$

Example 3.1. *Take* $f(x) = g(x) = x + 1$, $x_0 = 1$, $L_f = L_g = 2$. *Then*

$$\lim_{x\to 1}(x+1)^2 = 4\,.$$

Example 3.2. *For every polynomial* p *and every* x_0,

$$\lim_{x\to x_0} p(x) = p(x_0)\,.$$

4. More sophisticated results

The computation of some limits requires a degree of ingenuity.

4.1. **A famous trigonometric limit.** Let us show that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1\,.$$

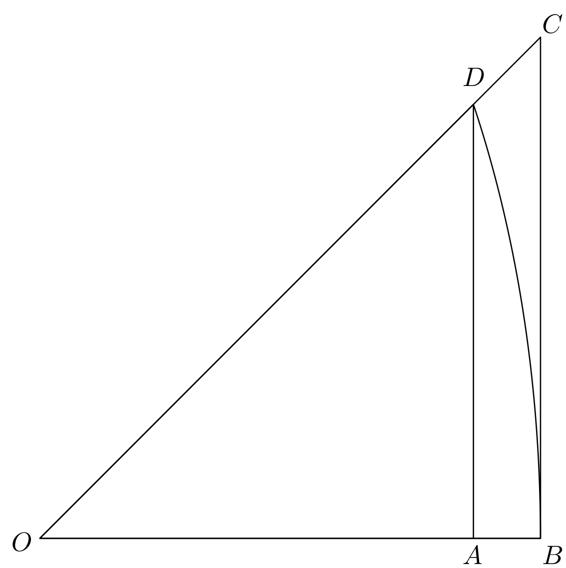

Figure 1. A sector of positive angle x (in radians) of the unit circle

Figure 1 shows a sector of positive angle x (in radians) of the unit circle. In other words, the length of the arc BD is precisely equal to x. We have

$$\text{Area of triangle } OBD \le \text{Area of sector} \le \text{Area of triangle } OBC\,.$$

So

$$\frac{\sin x}{2} \le \frac{x}{2} \le \frac{\tan x}{2}\,.$$

The first inequality yields

$$\frac{\sin x}{x} \le 1$$

and the second

$$\frac{\sin x}{x} \ge \cos x\,.$$

Combining these two results, we obtain

$$\cos x \le \frac{\sin x}{x} \le 1\,. \tag{4.1}$$

Now, by definition of $\cos x$,

$$\lim_{x\to 0}\cos x = 1\,.$$

Reporting this in (4.1), we find

$$\lim_{x\to 0+}\cos x = 1 \le \lim_{x\to 0+}\frac{\sin x}{x} \le 1\,.$$

The limit

$$\lim_{x\to 0-}\frac{\sin x}{x} = 1$$

can be obtained in the same way.

4.2. **The exponential function.** The exponential number e is defined by

$$\mathrm{e} := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\,.$$

That the limit on the right-hand side exists is a non-trivial fact (see Analysis), but if we accept it, then it is not too hard to show that, for every positive integer n, we have

$$\left(1+\frac{1}{n}\right)^n < \mathrm{e} < \left(1+\frac{1}{n}\right)^{n+1}\,.$$

This gives a method for computing the decimal expansion of e:

$$\mathrm{e} = 2.7182818\dots$$

Now let $x \in \mathbb{N}$. We have

$$\mathrm{e}^x = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} \underset{\substack{\uparrow\\ nx=m}}{=} \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m$$

$$\overset{\substack{m=n\\ \downarrow}}{=} \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\,.$$

We then *define* the exponential function on $\mathbb{R}$ by

$$\mathrm{e}^x := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\,.$$

Notation . We use

$$\mathrm{e}^x \quad \text{and} \quad \exp(x)$$

interchangeably.

Example 4.1. *Show that*

$$\lim_{x\to 0+}\frac{\mathrm{e}^x - 1}{x} = 1\,.$$

Solution*: Let $x > 0$ and $n \in \mathbb{N}$. Recall the binomial expansion*

$$\left(1+\frac{x}{n}\right)^n = 1 + \frac{n}{n}x + \frac{n(n-1)}{2n^2}x^2 + \cdots + \binom{n}{n}\frac{1}{n^n}x^n\,.$$

We deduce

$$1 + x \le \left(1+\frac{x}{n}\right)^n \le 1 + x + \cdots + x^n \le 1 + x + \cdots + x^n + \cdots$$

$$= \frac{1}{1-x} = \frac{1-x+x}{1-x} = 1 + \frac{x}{1-x}\,.$$

That is

$$1 + x \le \left(1 + \frac{x}{n}\right)^n \le 1 + \frac{x}{1-x}\,.$$

Let $n \to \infty$*:*

$$1 + x \le \mathrm{e}^x \le 1 + \frac{x}{1-x}\,.$$

Hence

$$1 \le \frac{\mathrm{e}^x - 1}{x} \le \frac{1}{1-x}\,.$$

Let $x \to 0+$*:*

$$1 \le \lim_{x \to 0+} \frac{\mathrm{e}^x - 1}{x} \le \lim_{x \to 0+} \frac{1}{1-x} = 1\,.$$

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.
2. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.