MATH11007 NOTES 3: THE DERIVATIVE

1. Motivation and definition

Given a curve of equation

$$y = f(x)\,,$$

find the equation of the tangent and normal lines at x_0.

(Partial) solution: let $y_0 = f(x_0)$. The equation of the tangent line is

$$\frac{y - y_0}{x - x_0} = m \quad \text{(tangent)}$$

where m is the *slope* of the tangent line. The equation of the normal line is

$$\frac{y - y_0}{x - x_0} = -1/m \quad \text{(normal)}\,.$$

Example 1.1. *Let the curve be the straight line of equation*

$$y = mx + b\,.$$

For every x_0, the tangent line is the curve itself, and its slope is given by the formula

$$m = \frac{f(x_0 + h) - f(x_0)}{h}$$

where h is an arbitrary number.

Example 1.2. *Next, consider the parabola of equation*

$$y = f(x) = ax^2 + b\,.$$

Let x_0 and $h \neq 0$ be two numbers. For h small, the ratio

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{a(x_0 + h)^2 + b - ax_0^2 - b}{h} = \frac{2ax_0 h + h^2}{h} = 2ax_0 + h$$

approximates the slope of the line tangent to the parabola at x_0. We obtain the exact value of the slope by letting h tend to 0:

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = 2ax_0\,.$$

Definition 1.1. Let $f : A \to B$ and $x_0 \in A$. We say that f is *differentiable at x_0* if the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. Further, if f is differentiable at every point of its domain, then we say that f is *differentiable*. In this case, the function $f' : A \to B$ defined by

$$f'(x) := \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\,,$$

is called the *derivative* of f.

Notation . We shall often use the following alternative notations for the derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{instead of} \quad f'\,,$$
$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \quad \text{instead of} \quad f''\,,$$
$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} \quad \text{instead of} \quad f^{(n)}\,.$$

We will see many other notations for the derivative in the course of your studies.

2. Derivative of the monomial

Let $n \in \mathbb{N}$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^n\,.$$

Let x_0, $h \neq 0$ be two real numbers. By using the binomial expansion, we can write

$$\begin{aligned}
\frac{f(x_0+h)-f(x_0)}{h} &= \frac{(x_0+h)^n - x_0^n}{h} \\
&= \frac{1}{h}\left[\binom{n}{0}x_0^n + \binom{n}{1}x_0^{n-1}h + \binom{n}{2}x_0^{n-2}h^2 + \cdots + \binom{n}{n}h^n - x_0^n\right] \\
&= \binom{n}{1}x_0^{n-1} + \binom{n}{2}x_0^{n-2}h + \cdots + \binom{n}{n}h^{n-1} \\
&\xrightarrow[h\to 0]{} \binom{n}{1}x_0^{n-1} = n x_0^{n-1}\,.
\end{aligned}$$

Hence

$$f'(x) = n x^{n-1}\,.$$

3. Derivative of the square root function

Let $f : (0,\infty) \to (0,\,\infty)$ be defined by

$$f(x) = \sqrt{x}\,.$$

Let $x_0,\ ,h > 0$. Then

$$\begin{aligned}
\frac{f(x_0+h)-f(x_0)}{h} &= \frac{\sqrt{x_0+h}-\sqrt{x_0}}{h} \\
&= \frac{\sqrt{x_0+h}-\sqrt{x_0}}{h}\left[\frac{\sqrt{x_0+h}+\sqrt{x_0}}{\sqrt{x_0+h}+\sqrt{x_0}}\right] = \frac{1}{h}\,\frac{\left(\sqrt{x_0+h}\right)^2 - \left(\sqrt{x_0}\right)^2}{\sqrt{x_0+h}+\sqrt{x_0}} \\
&= \frac{1}{h}\,\frac{x_0+h-x_0}{\sqrt{x_0+h}+\sqrt{x_0}} = \frac{1}{\sqrt{x_0+h}+\sqrt{x_0}} \xrightarrow[h\to 0]{} \frac{1}{2\sqrt{x_0}}\,.
\end{aligned}$$

Hence

$$f'(x) = \frac{1}{2\sqrt{x}}\,.$$

With some work, this result can be generalised as follows: for $f : (0,\infty) \to (0,\infty)$ given by

$$f(x) = x^\alpha\,, \quad \alpha \in \mathbb{R}\,,$$

we have

$$f'(x) = \alpha x^{\alpha-1}\,.$$

4. The exponential function

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \mathrm{e}^x$$

and let $x_0 \in \mathbb{R}$ and $h \neq 0$. Using the identity

$$\mathrm{e}^{a+b} = \mathrm{e}^a \mathrm{e}^b$$

and last week's result

$$\lim_{h \to 0} \frac{\mathrm{e}^h - 1}{h} = 1\,,$$

we easily obtain

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{\mathrm{e}^{x_0+h} - \mathrm{e}^{x_0}}{h} = \mathrm{e}^{x_0} \frac{\mathrm{e}^h - 1}{x} \xrightarrow[h \to 0]{} \mathrm{e}^{x_0}\,.$$

Hence

$$f'(x) = \mathrm{e}^x\,.$$

5. The sine and cosine functions

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sin x\,.$$

We shall use the identity

$$\sin(a+b) = \sin a \cos b + \sin b \cos a\,.$$

Let $x_0 \in \mathbb{R}$ and $h \neq 0$. Then

$$\begin{aligned}\frac{f(x_0+h) - f(x_0)}{h} = \frac{\sin(x_0+h) - \sin(x_0)}{h} &= \frac{\sin x_0 \cos h + \sin h \cos x_0 - \sin x_0}{h} \\ &= \underbrace{\sin x_0 \frac{\cos h - 1}{h}}_{A} + \underbrace{\cos x_0 \frac{\sin h}{h}}_{B}\,.\end{aligned}$$

Now

$$\begin{aligned}\frac{\cos h - 1}{h} &= \frac{\cos h - 1}{h} \frac{\cos h + 1}{\cos h + 1} = \frac{1}{h} \frac{\cos^2 h - 1}{\cos h + 1} \\ &= -\frac{1}{h} \frac{\sin^2 h}{\cos h + 1} = -\frac{\sin h}{h} \sin h \frac{1}{\cos h + 1} \xrightarrow[h \to 0]{} -1 \times 0 \times \frac{1}{2} = 0\,.\end{aligned}$$

Therefore

$$A \xrightarrow[h \to 0]{} 0\,.$$

On the other hand, from last week's lecture,

$$B \xrightarrow[h \to 0]{} \cos x_0\,.$$

Hence

$$f'(x) = \cos x\,.$$

A similar calculation gives

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x\,.$$

6. Some useful results

The following results follow immediately from the theorems or "rules" for limits.

Theorem 6.1. *Let f, $g : A \to B$ be differentiable. Then*

(1) *Sum rule: $f + g$ is differentiable and*
$$(f+g)' = f' + g' \,.$$
(2) *Product rule: fg is differentiable and*
$$(fg)' = f'g + fg' \,.$$
(3) *Quotient ule: if, in addition, g never vanishes, then f/g is differentiable and*
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \,.$$

Example 6.1. *By the product rule with $f = g = \sin x$,*
$$\frac{d}{dx}\sin^2 x = \cos x \sin x + \sin x \cos x = 2 \sin x \cos x \,.$$

Example 6.2. *By the quotient rule with $f = \sin x$ and $g = \cos x$,*
$$\frac{d}{dx}\tan x = \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \,.$$

7. The chain rule

Definition 7.1. Let $u : A \to B$ and $f : B \to C$. The *composition* $f \circ u : A \to C$ is the function defined by
$$(f \circ u)(x) = f\left(u(x)\right) \,.$$

Theorem 7.1 (Chain rule)**.** *Let $u : A \to B$ and $f : B \to C$ be differentiable. Then $f \circ u : A \to C$ is also differentiable and*
$$(f \circ u)'(x) := \frac{d}{dx} f\left(u(x)\right) = f'\left(u(x)\right) u'(x) \,.$$

Proof. Let $x_0 \in A$ and $h \neq 0$ be such that $x_0 + h \in A$. Using the trivial identity
$$u(x_0+h) = u(x_0) + h\frac{u(x_0+h) - u(x_0)}{h} \,,$$
we can write
$$\frac{f\left(u(x_0+h)\right) - f\left(u(x_0)\right)}{h} = \frac{f\left(u(x_0) + h\frac{u(x_0+h)-u(x_0)}{h}\right) - f\left(u(x_0)\right)}{h}$$
$$= \frac{\left[f\left(u(x_0) + h\frac{u(x_0+h)-u(x_0)}{h}\right) - f\left(u(x_0)\right)\right]\frac{u(x_0+h)-u(x_0)}{h}}{h\frac{u(x_0+h)-u(x_0)}{h}} \,.$$

Set
$$H := h\frac{u(x_0+h) - u(x_0)}{h}$$
and note that
$$\lim_{h\to 0} H = \lim_{h\to 0} h \lim_{h\to 0} \frac{u(x_0+h) - u(x_0)}{h} = 0 \times u'(x_0) = 0 \,.$$

Then, from the above,

$$\frac{f\left(u(x_0+h)\right)-f\left(u(x_0)\right)}{h}=\frac{f\left(u(x_0)+H\right)-f(u(x_0))}{H}\frac{u(x_0+h)-u(x_0)}{h}\,.$$

Hence

$$\lim_{h\to 0}\frac{f\left(u(x_0+h)\right)-f\left(u(x_0)\right)}{h}$$
$$=\lim_{H\to 0}\frac{f\left(u(x_0)+H\right)-f(u(x_0))}{H}\lim_{h\to 0}\frac{u(x_0+h)-u(x_0)}{h}$$
$$=f'(u(x_0))u'(x_0)\,.$$

□

Example 7.1 (Implicit differentiation)**.** *Take* $u(x)=\sqrt{x}$ *and* $f(x)=x^2$*. Then, for* $x>0$*,*

$$f(u(x))=x\,.$$

By the chain rule

$$1=\frac{d}{dx}x=\frac{d}{dx}f(u(x))=f'(u(x))u'(x)\,.$$

We deduce that

$$u'(x)=\frac{1}{f'(u(x))}=\frac{1}{2u(x)}=\frac{1}{2\sqrt{x}}\,.$$

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.
2. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.