MATH11007 NOTES 4: APPLICATIONS OF THE DERIVATIVE

1. Newton's method

Let f be a differentiable function and consider the equation

$$f(x) = 0\,. \tag{1.1}$$

In general, it is not possible to solve this equation exactly, but one can seek to *approximate* the solution. *Newton's method* starts with a guess, say x_0, and constructs a (hopefully) better guess as follows.

The equation of the tangent to the curve $y = f(x)$ at x_0 is

$$y = f(x_0) + (x - x_0)f'(x_0)\,.$$

The tangent line approximates the curve $y = f(x)$ very well in the neighbourhood of x_0. So the value, say x_1, where this tangent line intersects the horizontal axis should be a better guess for the solution of Equation (1.1). This yields the new guess

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}\,.$$

By iterating, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\,.$$

Under certain conditions, the sequence will converge to a zero of f.

Example 1.1. *To compute $\sqrt{2}$, we apply Newton's method to*

$$f(x) = x^2 - 2\,.$$

Then

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\,.$$

With the guess $x_0 = 1$, we find $x_1 = 3/2$, $x_2 = 17/12$ etc.

2. The angle between two intersecting curves

Suppose that the curves

$$y = f_1(x) \quad \text{and} \quad y = f_2(x)$$

intersect at (x_0, y_0). This happens if and only if

$$y_0 = f_1(x_0) = f_2(x_0)\,.$$

Let m_j be the slope of the tangent line of $y = f_j(x)$. Then

$$\theta_j := \arctan m_j$$

is the angle that the tangent line makes with the horizontal axis; see Figure 1. There are two angles between the tangent lines at the point of intersection. The sum of these two angles is π. Let φ be the angle such that

$$0 \le \varphi \le \pi/2\,.$$

We call this angle the *angle between the curves at the point of intersection.* In what follows, we explain how to calculate φ.

Geometrically,

$$\varphi = \theta_2 - \theta_1 \quad \text{or} \quad \theta_1 - \theta_2\,.$$

Hence

$$\begin{aligned}\tan\varphi = |\tan(\theta_2-\theta_1)| &= \left|\frac{\sin(\theta_2-\theta_1)}{\cos(\theta_2-\theta_1)}\right| = \left|\frac{\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1}{\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2}\right| \\ &= \left|\frac{\cos\theta_1\cos\theta_2}{\cos\theta_1\cos\theta_2}\,\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}\right| = \left|\frac{m_1-m_2}{1+m_1m_2}\right|\,.\end{aligned}$$

So we have the formula

$$\varphi = \arctan\left|\frac{m_1-m_2}{1+m_1m_2}\right|\,. \tag{2.1}$$

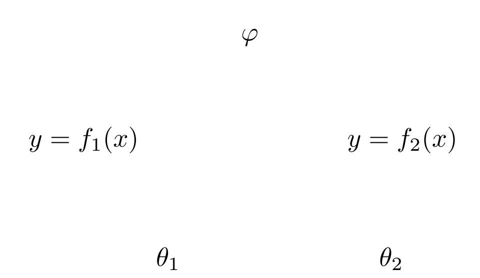

Figure 1. Two intersecting curves. The dashed lines are the tangent lines.

Example 2.1. *Let $f_1(x) = 1+(x+1)^2$ and $f_2(x) = 1+(x-1)^2$. The corresponding curves intersect at $x_0 = 0$. We have*

$$m_1 = f_1'(x_0) = 2 \quad \textit{and} \quad m_2 = f_2'(x_0) = -2\,.$$

The formula (2.1) yields

$$\varphi = \arctan(4/3) = 53.13^\circ\,.$$

3. L'Hospital's rule

Suppose that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \begin{cases} 0 & \text{or} \\ \infty & \text{or} \\ -\infty \end{cases}.$$

Then the value of the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is not immediately obvious! Now, suppose that f and g are differentiable. Then *L'Hospital's rule* asserts that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{3.1}$$

This is often useful in finding the limit on the left-hand side.

Example 3.1. *Let $f(x) = \sin x$ and $g(x) = x$. Then*

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$$

To determine the limit,

$$\lim_{x\to 0} \frac{\sin x}{x},$$

we apply L'Hospital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

This agrees with our earlier result, obtained by a different method.

Example 3.2. *Let $n \in \{\mathbb{N}\}$, $f(x) = x^n$ and $g(x) = e^x$. Then*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty.$$

To determine the limit,

$$\lim_{x\to\infty} \frac{x^n}{e^x},$$

we apply L'Hospital's rule n times

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

This shows that the exponential function grows faster than any power.

Let us give some hint of the proof of L'Hospital's rule in the particular case $a \in \mathbb{R}$ and

$$\lim_{x\to a} f(x) = f(a) = 0 = \lim_{x\to a} g(x) = g(a).$$

We then have (very sloppy!)

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$$

$$\lim_{x\to a} \left[\frac{f(x)-f(a)}{x-a} \frac{x-a}{g(x)-g(a)} \right] = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

$$= \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

4. Optimisation and curve sketching

Knowledge of the derivative(s) of a smooth function, say f, is very useful for sketching its graph. You will no doubt be familiar with the following facts:

(1) Points where f' is positive (respectively negative) are points of increase (respectively decrease) of f.
(2) Points where f' vanishes, i.e. *critical points* of f, are points of local extrema of f, or points of inflexion of f.
(3) The curve $y = f(x)$ is locally convex (respectively concave) at points where f'' is positive (respectively negative).

Another property (not related to differentiability) which is often useful in sketching a graph is the following:

Definition 4.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *even* if, for every $x \in \mathbb{R}$,

$$f(-x) = f(x)\,.$$

It is said to be *odd* if, for every $x \in \mathbb{R}$,

$$f(-x) = -f(x)\,.$$

Example 4.1. *The functions defined by the mappings*

$$x \mapsto x^2\,,\quad x \mapsto \cos(x)\,,\quad x \mapsto |x|$$

are all even. *The functions defined by the mappings*

$$x \mapsto x\,,\quad x \mapsto \sin(x)\,,\quad x \mapsto x\cos x$$

are all odd. *The function defined by the mapping*

$$x \mapsto x + x^2$$

is neither odd nor even.

Rather than discussing curve sketching in the abstract, we illustrate some useful tricks by means of the particular example

$$f(x) = \sin\left(\frac{6}{1+x^2}\right)\,.$$

(1) The function is even. So the graph is symmetric about the verical axis. We may therefore concentrate on the case $x \geq 0$, and then reflect.
(2) We have $f(0) = \sin 6 < \sin(2\pi) = 0$.
(3) We have

$$\lim_{x\to\infty} f(x) = \sin 0 = 0$$

and the limiting value is approached from above.
(4) Does f have zeroes? This requires

$$\frac{6}{1+x^2} = n\pi$$

for some natural number n. Only if $n = 1$ is there a real solution; it is $x = \sqrt{6/\pi - 1}$.

(5) What are the critical points? We compute the derivative by the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(\frac{6}{1+x^2}\right) = \cos\left(\frac{6}{1+x^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{6}{1+x^2}\right)$$
$$= \cos\left(\frac{6}{1+x^2}\right)\left(\frac{-6}{(1+x^2)^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(1+x^2\right) = \frac{-12x}{(1+x^2)^2}\cos\left(\frac{6}{1+x^2}\right).$$

So there are three critical points: in principle, we should compute f'' to determine the nature of each point, but the formula for the second derivative is too complicated, and it is simpler to examine the *sign* of f' instead.

(a) $x = 0$. We have $f'(x) < 0$ for x small and positive. So $x = 0$ is a local maximum.

(b) $x = \sqrt{4/\pi - 1}$. We have $f'(x) < 0$ for x slightly smaller and $f'(x) > 0$ fo x slightly larger than this critical point. It is therefore a local minimum.

(c) $x = \sqrt{12/\pi - 1}$. A local maximum since the function approaches its limiting value at ∞ from above.

See Figure 2 for the graph.

FIGURE 2. The graph of $y = \sin\left(\frac{6}{1+x^2}\right)$.

REFERENCES

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.