MATH11007 NOTES 5: THE FUNDAMENTAL THEOREM OF CALCULUS

1. The concept of primitive

Definition 1.1. We say that a function $F : A \to \mathbb{R}$ is a *primitive* (or an anti-derivative, or an indefinite integral) of the function $f : A \to \mathbb{R}$ if F is differentiable and

$$F' = f .$$

Example 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = x^n , \quad n \in \mathbb{N} .$$

Then the following functions are all primitives of f:

(1)
$$F(x) = \frac{x^{n+1}}{n+1} .$$

(2)
$$F(x) = \frac{x^{n+1}}{n+1} + 10^6 .$$

Example 1.2. *Let $f : (0, \infty) \to (0, \infty)$ be defined by*

$$f(x) = \sqrt{x} .$$

Then

$$F(x) = \frac{2}{3} x^{3/2} + 1$$

is a primitive.

Example 1.3. *Let $f(x) = \sin x$. Then $F(x) = -\cos x + \pi$ is a primitive.*

Example 1.4. *$F(x) = e^x + 1/2$ is a primitive of $f(x) = e^x$.*

Example 1.5. *$F(x) = \arctan x$ is a primitive of $f(x) = 1/(1 + x^2)$.*

Lemma 1.1. *If F_1 and F_2 are two primitives of the same function $f : A \to \mathbb{R}$, then there exists a constant c such that, for every $x \in A$,*

$$F_1(x) - F_2(x) = c .$$

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Proof. Let

$$G(x) = F_1(x) - F_2(x)\,.$$

Then, for every $x \in A$,

$$G'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0\,.$$

So, for every $x \in A$, the slope of the line tangent to the curve of equation

$$y = G(x)$$

is zero. It follows that the curve is a horizontal line, i.e. $y = c$ for some constant c. □

Given f and a function F, it is relatively easy to determine whether or not F is a primitive of f: it suffices to compute F'. But if we are given only f, it is not straightforward to *find* a primitive.

2. The area under a curve

Let $f : [a, b] \to \mathbb{R}$ be *continuous* and consider the curve of equation $y = f(x)$; see Figure 1.

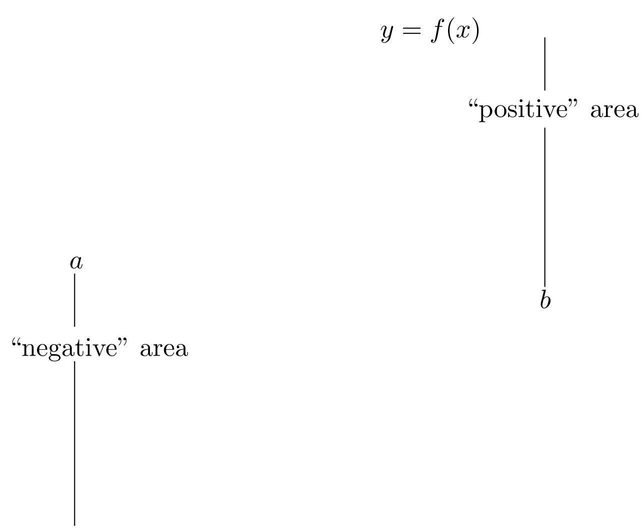

FIGURE 1. The area under a curve.

Notation .

$$\int_a^b f(x)\,\mathrm{d}x := \text{area under the curve between } x = a \text{ and } x = b\,.$$

Some remarks:

(1) It does not matter what letter one uses to denote the independent variable. Thus

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(u)\,\mathrm{d}u \quad \text{etc.}$$

(2) Geometrically, it is clear that, for every $a < c < b$,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x\,.$$

(3) We shall use the convention

$$\int_b^a f(x)\,\mathrm{d}x := -\int_a^b f(x)\,\mathrm{d}x\,.$$

Now, let a be fixed, but allow the right endpoint to vary between a and b: we obtain a function $I : [a,b] \to [0,\infty)$ defined by

$$I(x) := \int_a^x f(t)\,\mathrm{d}t\,. \tag{2.1}$$

This notation may be disconcerting at first sight, but remember that we agreed that we could use any letter for the independent variable, e.g.

$$\int_a^b f(x)\,\mathrm{d}x := \int_a^b f(u)\,\mathrm{d}u\,.$$

Then we are free to use x for the right endpoint of the interval! This explains the notation in Equation (2.1).

Theorem 2.1. *The function I is a primitive of f.*

Proof. Let $x_0 \in [a,b]$ and let $h \neq 0$ be such that $x_0 + h \in [a,b]$. We have

$$\frac{I(x_0+h)-I(x_0)}{h} = \frac{1}{h}\int_a^{x_0+h} f(t)\,\mathrm{d}t - \frac{1}{h}\int_a^{x_0} f(t)\,\mathrm{d}t$$
$$= \frac{1}{h}\int_a^{x_0} f(t)\,\mathrm{d}t + \frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,\mathrm{d}t - \frac{1}{h}\int_a^{x_0} f(t)\,\mathrm{d}t = \frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,\mathrm{d}t\,.$$

For simplicity, suppose that $h > 0$ and define

$$m_h(x_0) := \min_{x_0 \le t \le x_0+h} f(t) \quad \text{and} \quad M_h(x_0) := \max_{x_0 \le t \le x_0+h} f(t)\,.$$

It is then geometrically clear (see Figure 2) that

$$m_h(x_0)h \le \int_{x_0}^{x_0+h} f(t)\,\mathrm{d}t \le M_h(x_0)h\,.$$

After dividing by h, this gives

$$m_h(x_0) \le \frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,\mathrm{d}t \le M_h(x_0)\,. \tag{2.2}$$

Now, since, by hypothesis, f is continuous at x_0, we have

$$\lim_{h\to 0} m_h(x_0) = \lim_{h\to 0} M_h(x_0) = f(x_0)\,.$$

So we deduce from (2.2) that

$$\lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,\mathrm{d}t = f(x_0)\,.$$

In other words,

$$I'(x_0) = f(x_0)\,.$$

□

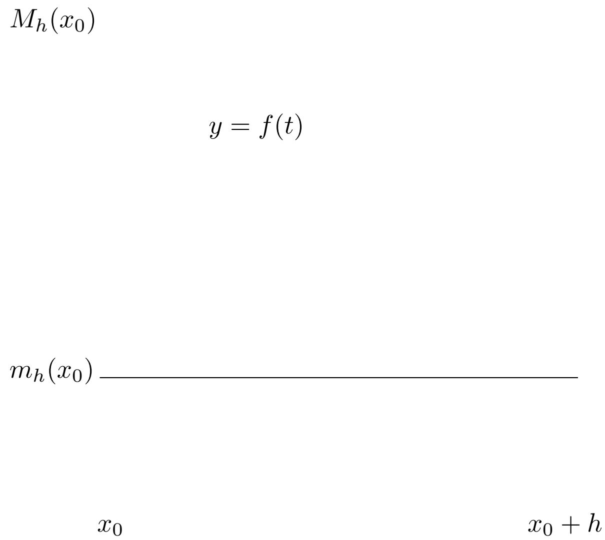

FIGURE 2. Proof of Theorem 2.1: the area under the curve is squeezed between $m_h(x_0)h$ and $M_h(x_0)h$.

Corollary 2.2. *Let F be any primitive of f. Then*

$$\int_a^b f(x)\, dx = F(b) - F(a)\,.$$

Proof. Let F be a primitive of f. Since I is also a primitive of f, there exists a constant c such that, for every $x \in [a, b]$,

$$I(x) = F(x) + c\,.$$

Then

$$\int_a^b f(x)\,\mathrm{d}x = I(b) = I(b) - I(a) = [F(b) + c] - [F(a) + c] = F(b) - F(a)\,.$$

□

Notation .

$$F(x) = \int f(x)\mathrm{d}x$$

means "F is a primitive of f". Also

$$F(x)\Big|_a^b := F(b) - F(a)\,.$$

Example 2.1. Problem*: Compute the area under the parabola of equation $y = x^2$ between $x = 0$ and $x = 1$.*

Solution*: Set $f(x) = x^2$. The function $F(x) = x^3/3$ is a primitive of f. So, by the Fundamental Theorem of Calculus, the area is*

$$\int_0^1 f(x)\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}\,.$$

Example 2.2. Problem*: Compute the area of a circle of unit radius.*
Solution*: Set* $f(x) = \sqrt{1-x^2}$. *Then the area is*

$$4\int_0^1 f(x)\,dx\,.$$

So we look for a primitive of f*!*

3. Substitutions

In this section, we consider a very useful technique for finding primitives.

Theorem 3.1. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous, and let* u *be a differentiable bijection with codomain* $[a,b]$. *Let* α *and* β *be such that*

$$u(\alpha) = a \quad \text{and} \quad u(\beta) = b\,.$$

Then

$$\int_a^b f(u)\,du = \int_\alpha^\beta (f\circ u)(x)u'(x)\,dx\,.$$

Proof. Let F be a primitive of f. Then, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x)\,.$$

In other words, $F(u(x))$ is a primitive of $(f\circ u)(x)u'(x)$. Hence, by the Fundamental Theorem of Calculus,

$$\int_a^b f(u)\,\mathrm{d}u = F(b) - F(a) = F(u(\beta)) - F(u(\alpha)) = \int_\alpha^\beta (f\circ u)(x)u'(x)\,\mathrm{d}x\,.$$

□

To illustrate the use of substitutions, let us go back to Example 2.2. We have

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x \overset{\underset{\downarrow}{u=x}}{=} \int_0^1 \sqrt{1-u^2}\,\mathrm{d}u$$

$$\overset{\underset{\downarrow}{u=\cos x}}{=} \int_{\frac{\pi}{2}}^0 \sqrt{1-\cos^2 x}\,(-\sin x)\ \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^2 x\,\mathrm{d}x\,.$$

Now, it is easy to find a primitive of $\sin^2 x$, and we obtain

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}\,.$$

So the area of a circle of unit radius is π.

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.