MATH11007 NOTES 6: MORE INTEGRATION

1. Integration by parts

Just as the chain rule for differentiation underlies the technique of *substitution*, so does the product rule for differentiation underly the technique of *integration by parts*.

Let u and v be two differentiable functions on some common domain. The product rule for differentiation (see Notes 3) asserts that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u(x)v(x)\right] = u(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x) + v(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\,.$$

So the product uv is a *primitive* of the function on the right-hand side of the above equality. Using the notation introduced previously (see Notes 5), we can write this as

$$u(x)v(x) = \int \left[u(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x) + v(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right]\,\mathrm{d}x\,.$$

After re-arrangement, this gives the familiar “integration by parts” formula:

$$\int u(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x)\,\mathrm{d}x = u(x)v(x) - \int v(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\,\mathrm{d}x\,. \tag{1.1}$$

Example 1.1. Find a primitive of $x\mathrm{e}^{-x}$. *We use Equation (1.1) with* $u(x) = x$ *and* $v(x) = -\mathrm{e}^{-x}$. *This yields*

$$\int x\mathrm{e}^{-x}\,\mathrm{d}x = -x\mathrm{e}^{-x} + \int \mathrm{e}^{-x}\,\mathrm{d}x = -(1+x)\mathrm{e}^{-x}\,.$$

Example 1.2. Compute

$$a_n := \int_0^{\pi/2} \sin^n x\,\mathrm{d}x\,, \quad n \in \mathbb{N}\,.$$

Solution: *By definition,*

$$a_0 = \int_0^{\pi/2} \mathrm{d}x = \frac{\pi}{2}$$

and

$$a_1 = \int_0^{\pi/2} \sin x\,\mathrm{d}x = -\cos x\Big|_0^{\pi/2} = 1\,.$$

Then, for $n \geq 0$,

$$a_{n+2} = \int_0^{\pi/2} \sin^{n+2} x \, \mathrm{d}x = \int_0^{\pi/2} \overbrace{\sin^{n+1} x}^{u} \, \underbrace{\sin x}_{v'} \, \mathrm{d}x$$
$$= -\sin^{n+1} x \cos x \Big|_0^{\pi/2} + (n+1) \int_0^{\pi/2} \sin^n x \cos x \cos x \, \mathrm{d}x$$
$$= 0 + (n+1) \int_0^{\pi/2} \sin^n x \left[1 - \sin^2 x\right] \mathrm{d}x = (n+1)\left(a_n - a_{n+2}\right) .$$

After re-arranging, we find

$$a_{n+2} = \frac{n+1}{n+2} \, a_n \, .$$

Hence

$$a_2 = \frac{1}{2} \, a_0 = \frac{\pi}{4} \, , \quad a_3 = \frac{2}{3} a_1 = \frac{2}{3} \, , \quad \textit{etc.}$$

2. Improper integrals

Definition 2.1. A non-empty set $S \subseteq \mathbb{R}$ is said to be *bounded* if there exists $c \in \mathbb{R}$ such that

$$\forall \, x \in S \, , \quad |x| \leq c \, .$$

A function $f : A \to B \subseteq \mathbb{R}$ is said to be bounded if the set

$$\{f(x) : x \in A\}$$

(i.e. the range of the function) is a bounded set. A set or a function that is not bounded is said to be *unbounded*.

Example 2.1. *The sets* $\mathbb{N}$ *and* $(1, \infty)$ *are unbounded. The sets* $[-1, 1]$ *and* $(0, 10^6)$ *are bounded. The function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \sin x$$

is bounded. The function $g : [1, 2] \to \mathbb{R}$ *defined by*

$$g(x) = \frac{1}{x}$$

is bounded, but if we change its domain to $(0, 2]$ *it becomes unbounded.*

Improper integrals are of two basic kinds:

(1) The integrand is continuous but the interval of integration is infinite. For example,
$$\int_0^{\infty} \mathrm{e}^{-x} \, \mathrm{d}x \quad \text{and} \quad \int_0^{\infty} \frac{\mathrm{d}x}{1+x^2}$$
are of this kind.

(2) The interval of integration is finite but the function is unbounded in the interval. For example
$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} \quad \text{and} \quad \int_{-1}^1 \frac{\mathrm{d}x}{x^2 - 1}$$
are of this kind.

We begin by considering integrals of the first kind.

Definition 2.2. Suppose that, for every $b > a$, the function $f : [a, \infty) \to \mathbb{R}$ is continuous in the interval $[a, b]$. Then

$$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x\,.$$

If the limit on the right exists, we say that the improper integral (on the left) is *convergent*; otherwise, we say that it is *divergent*.

Example 2.2. Determine whether the improper integral

$$\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x$$

is convergent or divergent.

Solution: *Let* $b > 0$. *We have*

$$\int_0^b \mathrm{e}^{-x}\,\mathrm{d}x = -\mathrm{e}^{-x}\Big|_0^b = 1 - \mathrm{e}^{-b} \xrightarrow[b\to\infty]{} 1\,.$$

Hence the integral is convergent. Its value is 1.

Example 2.3. Determine whether the improper integral

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^2}$$

is convergent or divergent.

Solution: *Let* $b > 0$. *We have*

$$\int_0^b \frac{\mathrm{d}x}{1+x^2} = \arctan x\Big|_0^b = \arctan b \xrightarrow[b\to\infty]{} \frac{\pi}{2}\,.$$

Hence the integral is convergent. Its value is $\pi/2$.

Example 2.4. Determine whether the improper integral

$$\int_0^\infty \frac{\mathrm{d}x}{1+\cos^2 x + x^4}$$

is convergent or divergent. This example illustrates the fact that it is not necessary to compute the value of the integral in order to obtain the answer.

Solution: *Let* $b > 1$. *We have*

$$\int_0^b \frac{\mathrm{d}x}{1+\cos^2 x + x^4} \le \int_0^b \frac{\mathrm{d}x}{1+x^4}$$

since $\cos^2 x \ge 0$. *Now*

$$\begin{aligned}\int_0^b \frac{\mathrm{d}x}{1+x^4} &= \int_0^1 \frac{\mathrm{d}x}{1+x^4} + \int_1^b \frac{\mathrm{d}x}{1+x^4} \\ &\le \int_0^1 \frac{\mathrm{d}x}{1+0} + \int_1^b \frac{\mathrm{d}x}{0+x^4} = 1 + \frac{-1}{3}x^{-3}\Big|_1^b \\ &= \frac{4}{3} + \frac{1}{3b^3} \xrightarrow[b\to\infty]{} \frac{4}{3}\,.\end{aligned}$$

This shows that, for every $b > 1$,

$$\int_0^b \frac{\mathrm{d}x}{1+\cos^2 x + x^4} \le \frac{2}{3}\,.$$

So the improper integral is convergent, even though we do not know how to compute its value.

Next, we consider the second kind of improper integral.

Definition 2.3. Let $a < b$ and suppose that, for every $\varepsilon > 0$, the unbounded function $f : (a,b] \to \mathbb{R}$ is continuous in $[a+\varepsilon, b]$. Then

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b f(x)\,\mathrm{d}x\,.$$

If the limit exists, we say that the integral is *convergent*; otherwise, we say that it is *divergent*.

Example 2.5. *Let* $0 < \alpha \neq 1$ *and consider*

$$\int_0^1 \frac{\mathrm{d}x}{x^\alpha}\,.$$

This integral is improper because the integrand is unbounded. Let $\varepsilon > 0$. *Then*

$$\int_\varepsilon^1 \frac{\mathrm{d}x}{x^\alpha} = \frac{x^{1-\alpha}}{1-\alpha}\Big|_\varepsilon^1 = \frac{1-\varepsilon^{1-\alpha}}{1-\alpha} \xrightarrow[\varepsilon \to 0+]{} \begin{cases} \frac{1}{1-\alpha} & \text{if } 0 < \alpha < 1 \\ \infty & \text{if } \alpha > 1 \end{cases}.$$

So the integral is convergent if $\alpha < 1$ *and divergent if* $\alpha > 1$. *What happens when* $\alpha = 1$*?*

Example 2.6. *Consider*

$$\int_0^1 \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x\,.$$

Let $\varepsilon > 0$. *Then*

$$\int_\varepsilon^1 \frac{\mathrm{e}^{-x}}{\sqrt{x}}\,\mathrm{d}x \le \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x}\Big|_\varepsilon^1 = 2\left(1-\sqrt{\varepsilon}\right) \xrightarrow[\varepsilon \to 0+]{} 2\,.$$

So the integral is convergent.

Example 2.7. *Consider*

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}}\,.$$

This integral is a hybrid of the two kinds. We write it as

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} + \int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}}\,.$$

The first integral is improper but convergent. The second is also improper. Let $b > 1$. *Then*

$$\int_1^b \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{x}\Big|_1^b = 2\left(\sqrt{b}-1\right) \xrightarrow[b \to \infty]{} \infty\,.$$

We conclude that the integral is divergent.

3. Taylor series

Let f be a "nice" function in the neighbourhood of $x = a$. The following statement is trivially true:

$$f \text{ is a primitive of } f'.$$

We can write this in the form

$$f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t\,. \tag{3.1}$$

Consider the integral on the right. Set

$$u(t) = f'(t) \quad \text{and} \quad v(t) = t - x\,.$$

Then

$$\begin{aligned} f(x) = f(a) + \int_a^x u(t)v'(t)\,\mathrm{d}t &= f(a) + u(t)v(t)\Big|_a^x - \int_a^x v(t)u'(t)\,\mathrm{d}t \\ &= f(a) + (t-x)f'(t)\Big|_a^x - \int_a^x (t-x)f''(t)\,\mathrm{d}t\,. \end{aligned}$$

That is:

$$f(x) = f(a) + (x-a)f'(a) - \int_a^x (t-x)f''(t)\,\mathrm{d}t\,. \tag{3.2}$$

We repeat this with the new integral on the right, i.e. we set

$$u(t) = f''(t) \quad \text{and} \quad v(t) = \frac{1}{2}(t-x)^2\,.$$

Then

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) - \int_a^x u(t)v'(t)\,\mathrm{d}t \\ &= f(a) + (x-a)f'(a) - u(t)v(t)\Big|_a^x + \int_a^x v(t)u'(t)\,\mathrm{d}t \\ &= f(a) + (x-a)f'(a) - \frac{1}{2}(t-x)^2 f''(t)\Big|_a^x + \int_a^x \frac{1}{2}(t-x)^2 f'''(t)\,\mathrm{d}t\,. \end{aligned}$$

Hence

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \int_a^x \frac{1}{2}(t-x)^2 f'''(t)\,\mathrm{d}t\,. \tag{3.3}$$

Continuing in this way, we find

$$f(x) = T_n(x) + R_n(x) \tag{3.4}$$

where

$$T_n(x) = f(a) + (x-a)f'(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) \tag{3.5}$$

and

$$R_n(x) = \int_a^x \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,\mathrm{d}t\,. \tag{3.6}$$

Some terminology:

- a is called the *point of expansion*.
- T_n is called the *Taylor polynomial* (of degree $\leq n$) of f when a is the point of expansion.

- R_n is called the *remainder.*

Example 3.1. *Let* $f(x) = \mathrm{e}^x$ *and* $a = 0$*. Then, for every* $j \in \mathbb{N}$*,*

$$f^{(j)}(x) = \mathrm{e}^x\,.$$

So

$$T_n(x) = \sum_{j=0}^{n} \frac{1}{j!} x^j$$

and

$$R_n(x) = \int_0^x \frac{1}{n!}(x-t)^n\,\mathrm{e}^t\,\mathrm{d}t\,.$$

One possible use of Equation (3.4) is the following: for x close to a, one expects the remainder $R_n(x)$ to be small. Hence the (possibly complicated) function f should be well approximated by the (simple) polynomial T_n in the neighbourhood of the point of expansion.

More daring is the following idea: let x be fixed. If the remainder $R_n(x)$ vanishes in the limit as $n \to \infty$, then

$$f(x) = \lim_{n\to\infty} T_n(x) = \sum_{j=0}^{\infty} \frac{1}{j!}(x-a)^j f^{(j)}(a)\,. \tag{3.7}$$

The series on the right is called the *Taylor series* of f when a is the point of expansion. Note that existence of the series is only guaranteed if f has derivatives of every order at a.

Example 3.2. *Let* $f(x) = \mathrm{e}^x$ *and* $a = 0$*. For simplicity, suppose that* $x > 0$*. Then*

$$\begin{aligned}|R_n(x)| = \left|\int_0^x \frac{1}{n!}(x-t)^n\,\mathrm{e}^t\,\mathrm{d}t\right| &\le \int_0^x \frac{1}{n!}|x-t|^n\,\mathrm{e}^t\,\mathrm{d}t \\ &\le \int_0^x \frac{1}{n!}x^n\,\mathrm{e}^x\,\mathrm{d}t = \frac{x^{n+1}}{n!}\mathrm{e}^x \xrightarrow[n\to\infty]{} 0\,.\end{aligned}$$

We deduce that

$$\mathrm{e}^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}\,.$$

From Euler's formula

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

we deduce

$$\cos x = \sum_{j=0}^{\infty}(-1)^j \frac{x^{2j}}{(2j)!} \quad \textit{and} \quad \sin x = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{x^{2j-1}}{(2j-1)!}$$

Example 3.3. *Let* $f(x) = 1/(1+x^2)$ *and* $a = 0$*. Then*

$$f'(x) = \frac{-2x}{(1+x^2)^2}\,,\quad f''(x) = \frac{-2}{(1+x^2)^2} + \frac{(2x)^2}{(1+x^2)^3}$$

and so on. We have

$$T_{2n}(x) = \sum_{j=0}^{n}(-1)^j x^{2j}\,.$$

The series

$$\sum_{j=0}^{\infty}(-1)^j x^{2j}$$

converges to $f(x)$ *for* $|x| < 1$ *but diverges for* $|x| > 1$.

References

1. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.