MATH11007 SHEET 1: BEFORE CALCULUS

Set on Friday, October 12: Questions 3, 5 and 7.

1. The real line; intervals and inequalities (see [1], Chapter 1)

(1) Solve the following inequalities:

$$\text{(a)}\ -3 < 4 - x < 8\ ; \quad \text{(b)}\ \frac{3x-1}{x} < 4\ ; \quad \text{(c)}\ \frac{x}{x+5} < 1\ ;$$
$$\text{(d)}\ \frac{2x-1}{3x+2} > 3\ ; \quad \text{(e)}\ \left|\frac{3x-1}{x}\right| > 4\ ; \quad \text{(f)}\ \left|\frac{x}{x+5}\right| \le 2\,.$$

(2) Solve the following:

$$\text{(a)}\ |3x-4| = 3\ ; \quad \text{(b)}\ |x+5| = 2\ ; \quad \text{(c)}\ |2x-3| = |3x-2|\ ;$$
$$\text{(d)}\ |x+2| = |x+3|\ ; \quad \text{(e)}\ |x+2| = 4x-1\ ; \quad \text{(f)}\ |x-1| < |3x+1|\ ;$$
$$\text{(g)}\ |3x+4| < |2x-1|\,.$$

2. Polynomials and roots (see [1], Chapter 6)

(3) Solve the following:

$$\text{(a)}\ |x^2-2| < 1\ ; \quad \text{(b)}\ |x^2-2| < 3\,.$$

(4) Let p and q be two polynomials given respectively by

$$p(x) = x^3 - x^2 + x - 1 \quad \text{and} \quad q(x) = x^4 + x^2 + 3\,.$$

Write down the following polynomials:

$$\text{(a)}\ p+q\ ; \quad \text{(b)}\ p-q\ ; \quad \text{(c)}\ pq\,.$$

What is the maximal domain of the function $y = p(x)/q(x)$?

(5) Which of the following are correct definitions of functions? Draw the graphs of those that are.

$$\text{(a)}\ f(x) = \begin{cases} (x-1)^2 & \text{for } x \le 0 \\ 1-x^2 & \text{for } x \ge 0 \end{cases}\ ; \quad \text{(b)}\ f(x) = \begin{cases} x^2 & \text{for } x \le 0 \\ 1-x^2 & \text{for } x \ge 0 \end{cases}\ ;$$
$$\text{(c)}\ f(x) = \frac{x}{1-x^2}\ \text{for } x \ge 0\ ; \quad \text{(d)}\ f(x) = \frac{x}{1-x^2}\ \text{for } x > 1\,.$$

(6) *After Euler (1749); see* [2], *p.14.* Given B, C and D, solve the equation

$$f(x) := x^4 + Bx^2 + Cx + D = 0$$

Help: Show that

$$f(x) = (x^2 + ux + \alpha)(x^2 - ux + \beta)$$

for some numbers u, α and β.

3. Trigonemetric functions (see [1], Chapter 16)

(7) Evaluate the following "by hand":

(a) $\tan(7\pi/4)$; (b) $\sec^2(\pi/5) - \tan^2(\pi/5)$; (c) $\csc(19\pi/6)$; (d) $\cos^2(\pi/8) - \sin^2(\pi/8)$.

(8) For $n \in \mathbb{N}$, define

$$T_n(x) := \cos(n \arccos x) \ .$$

(a) Show that T_0 and T_1 are polynomials of degree 0 and 1 respectively.
(b) Show that

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x) \, .$$

Deduce that T_n is a polynomial of degree n. These polynomials are called the *Chebyshev* polynomials.
(c) Find the roots of T_n.

(9) *After al-Kāshi (1429); see* [2], *p.47.* By using regular polygons and the half-angle formula, it is possible to obtain expressions for the sine of the angles $\pi/5$, $\pi/3$ and $\pi/60$ in terms of square roots. Let $x = \sin(\pi/180)$. Show that

$$-4x^3 + 3x = \sin(\pi/60) \, .$$

(10) You are given that

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

for every θ. Let $\theta = \sqrt{x} := \pi/6$. Then

$$\frac{\sqrt{3}}{2} - 1 = -\frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots \, .$$

By truncating the series on the right after one, two and three terms, find approximations of π.

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.
2. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.