MATH11007 SHEET 3: THE DERIVATIVE

Abstract. See [1], Chapters 9, 10 and 11, for similar problems.

Set on Friday, October 26: Questions 2, 3 and 4.

(1) Compute the derivative of the following functions.

(a) $(1-x^2)^2$; (b) $\sin^2 x$; (c) $\cos^2(x^3)$; (d) $x^2 \tan x$; (e) $\sqrt{1+x^3}$.

(2) The height h of the tower of Pisa on its lowest side is 55.86 meters. From there, Galilei drops a cannon ball. Given that the height, say $y(t)$, of the ball after t seconds is given by

$$y(t) = h - \frac{gt^2}{2} ,$$

where g is the gravitational constant, find the velocity at which the ball hits the ground.

(3) Find the slope of the tangent lines to the parabola of equation $y = -x^2 + 6x - 5$ at its points of intersection with the x-axis.

(4) Consider a cylinder of radius r and height h.
  (a) What are the volume, say V, and the surface area, say S, of the cylinder?
  (b) Compute the instantaneous rate of change of the volume as the radius varies while the surface area is held fixed.
  (c) Compute the instantaneous rate of change of the surface area as the radius varies while the volume is held fixed.

(5) Let $f(x) = 1/x$. Then

$$xf(x) = 1 .$$

  (a) Deduce $f'(x)$.
  (b) Find a general formula for $f^{(n)}(x)$, where $n \in \mathbb{N}$.

(6) The logarithmic function $f : \mathbb{R}_+ \to \mathbb{R}$ is defined implicitly by

$$\mathrm{e}^{f(x)} = x .$$

Thus, it is the inverse function of the exponential function, and is denoted by

$$f(x) = \ln x .$$

Use implicit differentiation to compute $f'(x)$.

(7) By restricting the domains of the trigonometric functions $\sin x$, $\cos x$, and $\tan x$ to suitable intervals, one can define the inverse trigonometric functions (a) $\arcsin x$, (b) $\arccos x$ and (c) $\arctan x$. Use implicit differentiation to determine the derivatives of these inverse trigonometric functions.

(8) (*After Euler, 1755* [2]). Let

$$f(x) = \mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}} .$$

Find $f'(x)$.

(9) Consider the circle of equation

$$x^2 + y^2 = r^2 . \tag{$\star$}$$

View y as a function (!) of x.

(a) Compute the derivative y' by two methods: (a) by implicit differentiation of Equation ($\star$) and (b) by solving Equation ($\star$) for y and then differentiating the resulting expression.

(b) Show that

$$\frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{r} .$$

For a general smooth curve in the (x, y) plane, the reciprocal of the expression on the left is called its *radius of curvature.*

(10) Let a, b, c, e, α, β, γ, η and ξ be fixed numbers. Use `Maple` to compute the derivative of the function $y(x)$ defined implicitly by

$$\frac{x}{y} + \frac{(a + bx)(c - x^2)}{(ex + \alpha x^2)^2} + ax\sqrt{\beta^2 + y^2} + \frac{y^2}{\sqrt{\gamma^2 + \eta x + \xi x^2}} = 0 .$$

This problem has no particular application or geometrical interpretation; Leibniz (1684) simply "cooked it up" in order to demonstrate the advantages of his calculus.

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.
2. E. Hairer and G. Wanner, *Analysis by its History*, Springer-Verlag, New-York, 1996.