MATH11007 SHEET 4: APPLICATIONS OF THE DERIVATIVE

Set on Friday, November 2: Qs 1, 2 and 6.

(1) The chains of the Clifton suspension bridge are attached to supporting towers that are $L = 214$ m apart. These chains hang with the lowest point $h = 21.3$ m below the point of suspension.

(a) Galilei (1638) asserted that a chain hanging from two nails forms a parabola. Using the same assumption for the bridge's chains, estimate the angle between the chain and the tower.

(b) Some 20 years later, Christiaan Huygens, then aged only 16, realised that Galilei's assumption must be wrong. Leibniz and Johann Bernoulli (1691) later worked out that the shape of a hanging chain is a *catenary*, i.e. a curve of equation

$$y = a\left[\cosh\frac{x}{a} - 1\right],$$

where

$$\cosh u := \frac{\mathrm{e}^u + \mathrm{e}^{-u}}{2}$$

is the *hyperbolic cosine* function.

(i) Use Newton's method with the initial guess $a_0 = L$ to find a such that

$$h = a\left[\cosh\frac{L}{2a} - 1\right].$$

(ii) What is the angle between the chain and the tower if we make the assumption that the shape of the bridge's chain is a catenary?

See
`http://en.wikipedia.org/wiki/Catenary`

(2) Show that the curves $y = x^3 + 2$ and $y = 2x^2 + 2$ have a common tangent line at the point $(0, 2)$ and intersect at the point $(2, 10)$ at an angle ϕ such that $\tan\phi = 4/97$.

(3) Show that the ellipse $4x^2 + 9y^2 = 45$ and the hyperbola $x^2 - 4y^2 = 5$ intersect at a right angle.

(4) Let a_i, $1 \le i \le n$ be given numbers. Find the value of x that minimises

$$y = \sum_{i=1}^{n} (x - a_i)^2.$$

(5) Let $a > 0$ and consider the parabola of equation $y = a^2 - x^2$. For $0 \leq x_0 \leq a$ and $y_0 = y(x_0)$, the tangent line at (x_0, y_0) and the coordinate axes determine a triangle that lies in the first quadrant. Find the value of x_0 that minimimes the triangle's area.

(6) Let $n \in \mathbb{Z}_+ = \{1, 2, \ldots\}$. By using L'Hospital's rule or any other method, find the following limits:

$$\text{(a)} \lim_{x\to n} \frac{x^n - n^n}{x-n}\,; \quad \text{(b)} \lim_{x\to -n} \frac{\ln(n+1+x)}{n+x}\,; \quad \text{(c)} \lim_{x\to 0} \frac{2^{nx}-2^x}{x}\,;$$

$$\text{(d)} \lim_{x\to 0} \frac{x^n \mathrm{e}^x}{1-\mathrm{e}^x}\,; \quad \text{(e)} \lim_{x\to\infty} \frac{5x + n\ln x}{x + n^2 \ln x}\,; \quad \text{(f)} \lim_{x\to\infty} \frac{x^n + x^2}{\mathrm{e}^x + 1}\,.$$

(7) For each of the following curves, find the critical points, determine their type, and hence sketch the graph.

$$\text{(a)}\ y = x^4 - 2x^2 + 1\,; \quad \text{(b)}\ y = \frac{x}{(3+x^2)^2}\,; \quad \text{(c)}\ y = 3x^4 - 8x^3 + 5\,;$$

$$\text{(d)}\ y = \frac{x}{1-x^2}\,; \quad \text{(e)}\ y = x + 4/x\,; \quad \text{(f)}\ y^2 + x(x-2) = 0\,;$$

$$\text{(g)}\ y^2 = x^3\,; \quad \text{(h)}\ y^3 = x^2\,; \quad \text{(i)}\ y = \frac{x}{\sqrt{x^2-1}}\,;$$

$$\text{(j)}\ y = \sin\left(\frac{6}{1+x^2}\right)\,; \quad \text{(k)}\ y = \sqrt{1-x} + \sqrt{x+2}\,; \quad \text{(l)}\ |x|^a + |y|^a = 1\,,\ \ a > 0\,.$$

(8) Let $a > 0$ and $p \in \mathbb{Z}_+$. How would one use Newton's method to compute

$$\text{(a)}\ a^{\frac{1}{p}}\,? \quad \text{(b)}\ \ln p\,?$$

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.
(a) Show that, if f is even, then f' is odd.
(b) Show that, if f is odd, then f' is even.

(10) A rectangular playpen of specified height and unit area is to be made. Let x and y be the lengths of the sides, and assume that an opening of length λx, where $0 < \lambda < 1$, is required in one of the sides of length x. Find the values of x and y such that the construction requires the least amount of material.

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.