MATH11007 SHEET 5: THE FUNDAMENTAL THEOREM OF CALCULUS

ABSTRACT. Similar questions can be found in [1], Chapters 22, 23 and 24.

Set on Friday, November 9: Qs 1, 2 and 3.

(1) The *hyperbolic functions* are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x},$$
$$\text{sech} x = \frac{1}{\cosh x}, \quad \text{cosech} x = \frac{1}{\sinh x}, \quad \coth x = \frac{1}{\tanh x}.$$

(a) Show that, for every $x \in \mathbb{R}$,

$$\cosh^2 x - \sinh^2 x = 1.$$

(b) For each of the hyperbolic functions, find (i) the derivative and (ii) a primitive.

(2) Evaluate the following.

$$\text{(a)}\ \psi'(x) \text{ where } \psi(x) = \int_x^1 \sin t^2 \, dt\,; \quad \text{(b)} \int \frac{\ln y}{y} dy\,; \quad \text{(c)} \int \frac{\sin\theta \, d\theta}{2\cos\theta - 5}\,;$$
$$\text{(d)} \int_1^2 p\, e^{p^2} dp\,; \quad \text{(e)} \int \frac{t\, dt}{t^2 - 5t + 6}\,; \quad \text{(f)} \int_2^4 \frac{du}{u(u-1)^2}\,;$$
$$\text{(g)} \int \frac{dx}{\sqrt{1 - 9x^2}}\,; \quad \text{(h)} \int_0^{\frac{\pi}{2}} \frac{\sin\beta \, d\beta}{\sqrt{4 - \cos^2\beta}}\,; \quad \text{(i)} \int \frac{d\eta}{\sqrt{3 + 2\eta - \eta^2}}\,;$$
$$\text{(j)} \int \frac{dx}{x^2(x^2+9)}\,; \quad \text{(k)} \int \frac{3t^2 - 1}{t^3 - t} dt\,; \quad \text{(l)} \int_0^{\pi} \frac{\sin\sqrt{\sigma}}{\sqrt{\sigma}} d\sigma\,.$$

(3) Let $a \in \mathbb{R}$. Evaluate

$$\int_a^{a^2} |x| \, dx\,.$$

(4) Is the following calculation correct?

$$\int_{-1}^{1} \frac{dx}{x^2} = \frac{-1}{x}\Big|_{-1}^{1} = -1 - 1 = -2\,.$$

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $L > 0$. Show that, if f is even, then

$$\int_{-L}^{L} f(x)\,\mathrm{d}x = 2\int_{0}^{L} f(x)\,\mathrm{d}x\,.$$

What can you say about the integral on the left if f is *odd*?

(6) A rectangle has two vertices on the x axis and two vertices on the curve $y = \mathrm{e}^{-x^2}$. Show that the rectangle's area is greatest when it has two vertices at points of inflection of the curve.

(7) Find the area bounded by the vertical lines $x = 1$ and $x = 2$ and the curves of equation $y = x^2$ and $y = x^4$ respectively.

(8) Consider the problem of computing the area under the curve of equation

$$y = x^p\,, \quad p > 0\,,$$

between the vertical lines $x = 0$ and $x = b$.

(a) Use the Fundamental Theorem of Calculus to find the area.

(b) This problem was solved by Fermat in 1636, i.e. before the invention of Calculus. In the following, we shall go through the steps of Fermat's method.

(i) To visualise the problem, draw the graph of $y = x^p$ for the particular case $p = 1/2$.

(ii) Choose $0 < h < 1$ and set

$$x_n = (1-h)^n b\,, \;\; y_n = x_n^p\,, \quad n \in \mathbb{N}\,.$$

Consider the rectangle with vertices $(x_{n+1},\, 0)$, $(x_n,\, 0)$, $(x_n,\, y_{n+1})$ and $(x_{n+1},\, y_{n+1})$ and call its area a_n. Also, call A_n the area of the rectangle with vertices $(x_{n+1},\, 0)$, $(x_n,\, 0)$, $(x_n,\, y_n)$ and $(x_{n+1},\, y_n)$. Show that

$$a_n \le \int_{x_{n+1}}^{x_n} x^p\,\mathrm{d}x \le A_n$$

and hence

$$s(h) \le \int_0^b x^p\,\mathrm{d}x \le S(h)\,,$$

where

$$s(h) := \sum_{n=0}^{\infty} a_n \quad \text{and} \quad S(h) := \sum_{n=0}^{\infty} A_n\,.$$

(iii) Next, without evaluating $s(h)$ and $S(h)$, show that

$$\lim_{h\to 0} s(h) = \lim_{h\to 0} S(h) = \int_0^b x^p\,\mathrm{d}x\,.$$

(iv) Hence, find the value of the integral by evaluating $S(h)$ and then taking the limit $h \to 0$.

References

1. Frank Ayres, Jr. and Elliott Mendelson, *Schaum's Outline of Calculus, Fourth Edition*, McGraw–Hill, 1999.