MATH11007 SHEET 6: MORE INTEGRATION

Set on Monday, November 16: Qs 1, 3 and 8.

(1) Compute the following integrals.

$$\text{(a)}\ \int_0^{\ln 3} \cosh(2u)\,\mathrm{d}u\ ;\quad \text{(b)}\ \int_0^1 \frac{\mathrm{d}x}{9-x^2}\ ;\quad \text{(c)}\ \int s\sqrt{2-s}\,\mathrm{d}s\ ;$$

$$\text{(d)}\ \int x^2 \mathrm{e}^{2x}\,\mathrm{d}x\ ;\quad \text{(e)}\ \int_0^1 \xi\sinh\xi\,\mathrm{d}\xi\ ;\quad \text{(f)}\ \int \cos^3\theta\sin^2\theta\,\mathrm{d}\theta\ ;$$

$$\text{(g)}\ \int_0^{\frac{\pi}{2}} \frac{\cos\varphi}{1+\sin^2\varphi}\,\mathrm{d}\varphi\ ;\quad \text{(h)}\ \int_{-6}^{-2} \frac{\mathrm{d}y}{y^2+8y+20}\ ;\quad \text{(i)}\ \int_0^1 \frac{\mathrm{d}p}{2-\sqrt[3]{p}}\ ;$$

$$\text{(j)}\ \int \frac{\mathrm{d}x}{x(x^2+1)}\ ;\quad \text{(k)}\ \int \frac{\mathrm{d}x}{x(x^2-1)}\ ;\quad \text{(l)}\ \int \frac{\mathrm{d}x}{x(x+1)^2}\ .$$

(2) Let

$$a_n = \int_0^{\frac{\pi}{2}} \cos^n\theta\,\mathrm{d}\theta\,,\quad n\in\mathbb{N}\,.$$

(a) Compute a_0 and a_1.
(b) Use integration by parts to express a_{n+2} in terms of a_n.
(c) Deduce a formula for a_n when n is even, and another formula when n is odd.

(3) Explain why the following integrals are improper and, in each case, determine whether the integral is convergent or divergent.

$$\text{(a)}\ \int_0^{\infty} \frac{\mathrm{d}x}{1+x^2}\ ;\quad \text{(b)}\ \int_1^{\infty} \frac{x}{1+x^2}\mathrm{d}x\ ;\quad \text{(c)}\ \int_0^{\frac{\pi}{2}} \sec x\,\mathrm{d}x\ ;$$

$$\text{(d)}\ \int_0^2 \frac{\mathrm{d}x}{x^{\frac{3}{2}}}\ ;\quad \text{(e)}\ \int_0^{\infty} \frac{\mathrm{d}x}{\sqrt{x}(1+x)}\ ;\quad \text{(f)}\ \int_0^{\infty} \frac{\sin^2 x}{1+x^2}\,\mathrm{d}x\ .$$

(4) Let

$$a_n = \int_0^{\infty} x^n\,\mathrm{e}^{-x}\,\mathrm{d}x\,,\quad n\in\mathbb{N}\,.$$

(a) Show that

$$a_{n+1} = (n+1)a_n\,.$$

(b) Deduce that

$$a_n = n!\,.$$

(5) Let a, $b > 0$. By differentiating with respect to a, compute

$$\int_0^\infty \frac{\mathrm{e}^{-bx} - \mathrm{e}^{-ax}}{x}\, \mathrm{d}x\,.$$

(6) For each of the following functions, find the Taylor polynomial T_3 when $x = 0$ is the point of expansion.

$$\text{(a)}\ \ \sin(2x)\,; \quad \text{(b)}\ \ \ln(1-x)\,; \quad \text{(c)}\ \ \mathrm{e}^{-x/2}\,.$$

(7) Find the Taylor polynomial T_n of f when $x = a$ is the point of expansion and

(a)

$$f(x) = \mathrm{e}^x \sin x\,,\ a = 0\,,\ n = 3$$

(b)

$$f(x) = \frac{1}{\sqrt[3]{x}}\,,\ a = 8\,,\ n = 2$$

(c)

$$f(x) = \sin x\,,\ a = \frac{\pi}{6}\,,\ n = 3$$

(8) (a) Let $n \in \mathbb{N}$ and, for $k = 0, 1, \ldots, n$, set

$$a_k = \int_0^1 x^n \left(\ln x\right)^k\, \mathrm{d}x\,.$$

Show that

$$a_k = \frac{(-1)^k k!}{(n+1)^{k+1}}\,.$$

(b) Following Johann Bernoulli (1697), prove that

$$\int_0^1 x^x\, \mathrm{d}x = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots\,.$$

Help: Write $x^x = \mathrm{e}^{x \ln x}$ and use Taylor's series for the exponential function.

(9) Find the Taylor series of

$$F(x) = \int_0^x f(t)\, \mathrm{d}t$$

where $f(t)$ is

$$\text{(a)}\ \ \frac{1}{\sqrt{1+t^3}}\,; \quad \text{(b)}\ \ \frac{1}{1+t^3}\,; \quad \text{(c)}\ \ \ln\cos t\,.$$

and the point of expansion is 0. Can you find a simple expression for F? Try `Maple`.