MATH11007 SOLUTION 1: BEFORE CALCULUS

(1) $x \in$
  (a) $(-4, 7)$.
  (b) $(-\infty, -1) \cup (0, \infty)$.
  (c) $(-5, \infty)$.
  (d) $(-1, -2/3)$.
  (e) $(-1, 0) \cup (0, 1/7)$.
  (f) $(-\infty, -10] \cup [-10/3, \infty)$.

(2) $x \in$
  (a) $\{1/3, 7/3\}$.
  (b) $\{-7, -3\}$.
  (c) $\{-1, 1\}$.
  (d) $\{-5/2\}$.
  (e) $\{1\}$.
  (f) $(-\infty, -1) \cup (0, \infty)$.
  (g) $(-5, -3/5)$.

(3) $x \in$
  (a) $(-\sqrt{3}, -1) \cup (1, \sqrt{3})$
  (b) $(-\sqrt{5}, \sqrt{5})$

(4) The maximal domain of p/q is $\mathbb{R}$.
  (a) $(p + q)(x) = x^4 + x^3 + x + 2$
  (b) $(p - q)(x) = -x^4 + x^3 - 2x^2 + x - 4$
  (c) $pq(x) = x^7 - x^6 + 2x^5 - 2x^4 + 4x^3 - 4x^2 + 3x - 3$

(5)
- OK. Use the commands
```
> f := x -> piecewise(x<0,(x-1)^2,x>=0,1-x^2);
> plot(f(x),x);
```
- No good; ambiguity at $x = 0$.
- No good; undefined at $x = 1$.
- OK. Use the commands
```
> f := x -> x/(x^2-1);
> plot(f(x),x=1..infinity);
```

(6) The hint yields the following equations for u, α and β:

$$\alpha + \beta - u^2 = B\,, \quad u(\beta - \alpha) = C \quad \text{and} \quad \alpha\beta = D\,.$$

The first and second equations can be written as

$$\beta + \alpha = B + u^2 \quad \text{and} \quad \beta - \alpha = \frac{C}{u}\,.$$

So we can express α and β in terms of u:

$$2\alpha = B + u^2 - \frac{C}{u} \quad \text{and} \quad 2\beta = B + u^2 + \frac{C}{u}\,.$$

Then, multiplying these two equations and using $\alpha\beta = D$, we find that u^2 solves the cubic equation

$$z^3 + 2Bz^2 + (B^2 - 4D)z - C^2 = 0\,.$$

So we have reduced the solution of the original quartic equation to that of a cubic equation (for which we have a "handy" formula).

(7)

$$\text{(a)}\ -1\,; \quad \text{(b)}\ 1\,; \quad \text{(c)}\ -2\,; \quad \text{(d)}\ 1/\sqrt{2}\,.$$

(8) We set $\theta = \arccos x$.

(a) $T_0(x) = \cos(0\cdot\theta) = 1$ and $T_1(x) = \cos(1\cdot\theta) = x$.

(b) We have

$$\begin{aligned} T_{n+1}(x) + T_{n-1}(x) &= \cos([n+1]\theta) + \cos([n-1]\theta) \\ &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin\theta + \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin\theta \\ &= 2\cos(n\theta)\cos\theta = 2T_n(x)x\,. \end{aligned}$$

By (a), for $n = 0$ and $n = 1$, T_n is a polynomial of degree n. Make the induction hypothesis. Then, by (b),

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

and so T_{n+1} is a polynomial of degree $n+1$.

(c) The roots are

$$\left\{\cos\left([2k-1]\frac{\pi}{2n}\right) \ :\ k = 1, \ldots, n\right\}\,.$$

(9) Let $\theta = \pi/180$ and $x = \sin\theta$. Then

$$\begin{aligned} \sin(\pi/60) = \sin(3\theta) &= \sin\theta\cos(2\theta) + \sin(2\theta)\cos\theta \\ &= \sin\theta\left\{\cos^2\theta - \sin^2\theta + 2\cos^2\theta\right\} = \sin\theta\left\{3\cos^2\theta - \sin^2\theta\right\} \\ &= \sin\theta\left\{3 - 4\sin^2\theta\right\} = 3x - 4x^3\,. \end{aligned}$$

(10) Truncating after one term, we obtain

$$x = 2 - \sqrt{3}\,. \quad \text{Hence}\ \ \pi \approx 6\sqrt{2-\sqrt{3}} = 3.1058\ldots\,.$$

After two terms:

$$x^2 - 12x + 24 - 12\sqrt{3} = 0\,. \quad \text{Hence}\ \ \pi \approx 3.14193\ldots\,.$$

After three terms:

$$x^3 - 30x^2 + 360x + 360\sqrt{3} - 720 = 0\,. \quad \text{Hence}\ \ \pi \approx 3.1415909\ldots\,.$$