

MATH11007 SOLUTION 2: LIMITS

(1)

(a) -3 ; (b) 2 ; (c) $\frac{1}{8}$; (d) 0 ; (e) $\frac{1}{4}$; (f) 0 ;

$$(g) \frac{x^2 + 5x + 4}{x^2 + 4x + 3} = \frac{(x+4)(x+1)}{(x+3)(x+1)} = \frac{x+4}{x+3} \xrightarrow{x \rightarrow -1} \frac{3}{2};$$

(h) $\frac{1}{6}$; (i) 0 ; (j) ∞ ; (k) $4x^3$;

$$(l) \frac{x-2}{\sqrt{x^2+5}-3} = \frac{x-2}{\sqrt{x^2+5}-3} \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} \\ = \frac{x-2}{x^2-4} (\sqrt{x^2+5}+3) = \frac{\sqrt{x^2+5}+3}{x+2} \xrightarrow{x \rightarrow 2} \frac{3}{2}.$$

(2)

(a) $-\frac{8}{3}$; (b) $-\infty$; (c) 15 ; (d) ∞ ; (e) ∞ ; (f) ∞ .

(3) (a)

$$\frac{\sqrt{x+3}}{2x-5} = \frac{\sqrt{x}\sqrt{1+3/x}}{x(2-5/x)} = \frac{1}{\sqrt{x}} \frac{1+3/x}{2-5/x}.$$

So

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x+3}}{2x-5} = \left(\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} \right) \left(\lim_{x \rightarrow \infty} \frac{1+3/x}{2-5/x} \right) \\ = \left(\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} \right) \frac{\lim_{x \rightarrow \infty} (1+3/x)}{\lim_{x \rightarrow \infty} (2-5/x)} = 0 \cdot \frac{1}{2} = 0.$$

(b) For $x > 0$,

$$\frac{2\sqrt{x^2+6}}{6+x} = \frac{2x\sqrt{1+2/x^2}}{x(1+6/x)} = 2 \frac{\sqrt{1+2/x^2}}{1+6/x} \xrightarrow{x \rightarrow \infty} 2.$$

(c) 0 .

(d)

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x}{e^x} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} \xrightarrow{x \rightarrow \infty} \frac{1 - 0}{1 + 0} = 1.$$

(e)

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{-x} e^{2x} - 1}{e^{-x} e^{2x} + 1} = \frac{e^{2x} - 1}{e^{2x} + 1} \xrightarrow{x \rightarrow -\infty} \frac{0 - 1}{0 + 1} = -1.$$

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(f) No limit.

(g) The limit is 0:

$$\left| \frac{\cos x}{x} \right| \leq \frac{1}{|x|} \xrightarrow{x \rightarrow \infty} 0.$$

(h) No limit.

(i) The limit is 0: For $x > 0$,

$$\left| \frac{\sin x}{\sqrt{x}} \right| \leq \frac{1}{\sqrt{x}} \xrightarrow{x \rightarrow \infty} 0.$$

(j)

$$\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt{x}} = \lim_{x \rightarrow 0} \left(\sqrt{x} \frac{\sin x}{x} \right) = \left(\lim_{x \rightarrow 0} \sqrt{x} \right) \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) = 0 \cdot 1 = 0.$$

(4) (a)

$$\frac{e^{2+h} - e^2}{h} = e^2 \frac{e^h - 1}{h} \xrightarrow{h \rightarrow 0} e^2 \cdot 1.$$

(b)

$$\begin{aligned} \frac{\cos(a+h) - \cos a}{h} &= \frac{\cos a \cos h - \sin a \sin h - \cos a}{h} \\ &= \cos a \frac{\cos h - 1}{h} - \sin a \frac{\sin h}{h} = \cos a \frac{\cos^2 h - 1}{h(\cos h + 1)} - \sin a \frac{\sin h}{h} \\ &= \frac{\sin h}{h} \left[-\frac{\cos a \sin h}{\cos h + 1} - \sin a \right] \xrightarrow{h \rightarrow 0} 1 \cdot \left[-\frac{\cos a \cdot 0}{1+1} - \sin a \right] = -\sin a. \end{aligned}$$

(c)

$$\begin{aligned} \frac{\sin(a+h) - \sin a}{h} &= \frac{\sin a \cos h + \sin h \cos a - \sin a}{h} = \sin a \frac{\cos h - 1}{h} + \frac{\sin h}{h} \cos a \\ &= -\sin a \frac{\sin h}{h} \frac{\sin h}{\cos h + 1} + \frac{\sin h}{h} \cos a \xrightarrow{h \rightarrow 0} -\sin a \cdot 1 \cdot \frac{0}{1+1} + 1 \cdot \cos a = \cos a. \end{aligned}$$

(d)

$$\frac{\sqrt{1+h} - 1}{h} = \frac{\sqrt{1+h} - 1}{h} \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} = \frac{1+h-1}{h} \frac{1}{\sqrt{1+h} + 1} \xrightarrow{h \rightarrow 0} 1 \cdot \frac{1}{1+1} = \frac{1}{2}.$$

(e) For h small enough

$$\frac{|1+h| - |1|}{h} = \frac{(1+h) - 1}{h} = 1 \xrightarrow{h \rightarrow 0} 1.$$

(f) For h small enough

$$\frac{|-1+h| - |-1|}{h} = \frac{-(-1+h) - (-(-1))}{h} = -1 \xrightarrow{h \rightarrow 0} -1.$$

(g)

$$\begin{aligned} \frac{1}{h} \left[\frac{\sin(a+h)}{\cos(a+h)} - \frac{\sin a}{\cos a} \right] &= \frac{\sin(a+h) \cos a - \sin a \cos(a+h)}{h \cos(a+h) \cos a} \\ &= \frac{\sin(a+h-a)}{h \cos(a+h) \cos a} = \frac{\sin h}{h} \frac{1}{\cos(a+h) \cos a} \xrightarrow{h \rightarrow 0} 1 \cdot \frac{1}{\cos a \cdot \cos a} = \sec^2 a. \end{aligned}$$

(h)

$$\frac{\sin^2(a+h) - \sin^2 a}{h} = \frac{\sin(a+h) - \sin a}{h} [\sin(a+h) + \sin a]$$
$$\xrightarrow{h \rightarrow 0} \cos a \cdot [\sin a + \sin a] = 2 \sin a \cos a.$$

(5) (a) Use the Maple commands

```
> f := x -> piecewise(x<0,x+1,x>=0,x);  
> plot(f(x),x=-2..2);
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(b)

$$\lim_{x \rightarrow 0^-} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = 0.$$

(c) No; it is discontinuous at $x = 0$.