MATH11007 SOLUTION 2: LIMITS

(1)

(a) -3 ; (b) 2 ; (c) $\frac{1}{8}$; (d) 0 ; (e) $\frac{1}{4}$; (f) 0 ;

(g) $$\frac{x^2+5x+4}{x^2+4x+3} = \frac{(x+4)(x+1)}{(x+3)(x+1)} = \frac{x+4}{x+3} \xrightarrow[x\to -1]{} \frac{3}{2} \,;$$

(h) $\frac{1}{6}$; (i) 0 ; (j) ∞ ; (k) $4x^3$;

(l) $$\frac{x-2}{\sqrt{x^2+5}-3} = \frac{x-2}{\sqrt{x^2+5}-3}\,\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}$$
$$= \frac{x-2}{x^2-4}\left(\sqrt{x^2+5}+3\right) = \frac{\sqrt{x^2+5}+3}{x+2} \xrightarrow[x\to 2]{} \frac{3}{2}\,.$$

(2)

(a) $-\frac{8}{3}$; (b) $-\infty$; (c) 15 ; (d) ∞ ; (e) ∞ ; (f) ∞ .

(3) (a)
$$\frac{\sqrt{x+3}}{2x-5} = \frac{\sqrt{x}\sqrt{1+3/x}}{x(2-5/x)} = \frac{1}{\sqrt{x}}\,\frac{1+3/x}{2-5/x}\,.$$

So
$$\lim_{x\to\infty}\frac{\sqrt{x+3}}{2x-5} = \left(\lim_{x\to\infty}\frac{1}{\sqrt{x}}\right)\left(\lim_{x\to\infty}\frac{1+3/x}{2-5/x}\right)$$
$$\left(\lim_{x\to\infty}\frac{1}{\sqrt{x}}\right)\frac{\lim_{x\to\infty}(1+3/x)}{\lim_{x\to\infty}(2-5/x)} = 0\cdot\frac{1}{2} = 0\,.$$

(b) For $x>0$,
$$\frac{2\sqrt{x^2+6}}{6+x} = \frac{2x\sqrt{1+2/x^2}}{x(1+6/x)} = 2\frac{\sqrt{1+2/x^2}}{1+6/x} \xrightarrow[x\to\infty]{} 2\,.$$

(c) 0.

(d)
$$\frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{e^x}{e^x}\,\frac{1-e^{-2x}}{1+e^{-2x}} = \frac{1-e^{-2x}}{1+e^{-2x}} \xrightarrow[x\to\infty]{} \frac{1-0}{1+0} = 1\,.$$

(e)
$$\frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{e^{-x}}{e^{-x}}\,\frac{e^{2x}-1}{e^{2x}+1} = \frac{e^{2x}-1}{e^{2x}+1} \xrightarrow[x\to -\infty]{} \frac{0-1}{0+1} = -1\,.$$

(f) No limit.

(g) The limit is 0:

$$\left|\frac{\cos x}{x}\right| \le \frac{1}{|x|} \xrightarrow[x\to\infty]{} 0\,.$$

(h) No limit.

(i) The limit is 0: For $x > 0$,

$$\left|\frac{\sin x}{\sqrt{x}}\right| \le \frac{1}{\sqrt{x}} \xrightarrow[x\to\infty]{} 0\,.$$

(j)

$$\lim_{x\to 0}\frac{\sin x}{\sqrt{x}} = \lim_{x\to 0}\left(\sqrt{x}\frac{\sin x}{x}\right) = \left(\lim_{x\to 0}\sqrt{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right) = 0\cdot 1 = 0\,.$$

(4) (a)

$$\frac{\mathrm{e}^{2+h}-\mathrm{e}^2}{h} = \mathrm{e}^2\frac{\mathrm{e}^h-1}{h} \xrightarrow[h\to 0]{} \mathrm{e}^2\cdot 1\,.$$

(b)

$$\begin{aligned}\frac{\cos(a+h)-\cos a}{h} &= \frac{\cos a\cos h-\sin a\sin h-\cos a}{h}\\ &= \cos a\frac{\cos h-1}{h} - \sin a\frac{\sin h}{h} = \cos a\frac{\cos^2 h-1}{h(\cos h+1)} - \sin a\frac{\sin h}{h}\\ &= \frac{\sin h}{h}\left[-\frac{\cos a\sin h}{\cos h+1}-\sin a\right] \xrightarrow[h\to 0]{} 1\cdot\left[-\frac{\cos a\cdot 0}{1+1}-\sin a\right] = -\sin a\,.\end{aligned}$$

(c)

$$\frac{\sin(a+h)-\sin a}{h} = \frac{\sin a\cos h+\sin h\cos a-\sin a}{h} = \sin a\frac{\cos h-1}{h} + \frac{\sin h}{h}\cos a$$

$$= -\sin a\frac{\sin h}{h}\frac{\sin h}{\cos h+1} + \frac{\sin h}{h}\cos a \xrightarrow[h\to 0]{} -\sin a\cdot 1\cdot\frac{0}{1+1} + 1\cdot\cos a = \cos a\,.$$

(d)

$$\frac{\sqrt{1+h}-1}{h} = \frac{\sqrt{1+h}-1}{h}\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} = \frac{1+h-1}{h}\frac{1}{\sqrt{1+h}+1} \xrightarrow[h\to 0]{} 1\cdot\frac{1}{1+1} = \frac{1}{2}\,.$$

(e) For h small enough

$$\frac{|1+h|-|1|}{h} = \frac{(1+h)-1}{h} = 1 \xrightarrow[h\to 0]{} 1\,.$$

(f) For h small enough

$$\frac{|-1+h|-|-1|}{h} = \frac{-(-1+h)-(-(-1))}{h} = -1 \xrightarrow[h\to 0]{} -1\,.$$

(g)

$$\begin{aligned}\frac{1}{h}\left[\frac{\sin(a+h)}{\cos(a+h)}-\frac{\sin a}{\cos a}\right] &= \frac{\sin(a+h)\cos a-\sin a\cos(a+h)}{h\cos(a+h)\cos a}\\ &= \frac{\sin(a+h-a)}{h\cos(a+h)\cos a} = \frac{\sin h}{h}\frac{1}{\cos(a+h)\cos a} \xrightarrow[h\to 0]{} 1\cdot\frac{1}{\cos a\cdot\cos a} = \sec^2 a\,.\end{aligned}$$

(h)

$$\frac{\sin^2(a+h)-\sin^2 a}{h} = \frac{\sin(a+h)-\sin a}{h}\,[\sin(a+h)+\sin a]$$
$$\xrightarrow[h\to 0]{} \cos a \cdot [\sin a + \sin a] = 2\sin a \cos a\,.$$

(5) (a) Use the `Maple` commands

```
> f := x -> piecewise(x<0,x+1,x>=0,x);
> plot(f(x),x=-2..2);
```

(b)

$$\lim_{x\to 0-} f(x) = 1 \quad \text{and} \quad \lim_{x\to 0+} f(x) = 0\,.$$

(c) No; it is disconuous at $x = 0$.