MATH11007 SOLUTION 3: THE DERIVATIVE

(1)

(a) $-4x(1-x^2)$; (b) $2\sin x\cos x$; (c) $-3x^2\sin(2x^3)$;

(d) $2x\tan x + x^2\sec^2 x$; (e) $\dfrac{3x^2}{2\sqrt{1+x^3}}$.

(2) The ball hits the ground when $y = 0$, i.e. when $t = \sqrt{2h/g}$; its velocity is then

$$y'(t) = -gt = -g\sqrt{2h/g} = -\sqrt{2hg} = -\sqrt{19.6\,\mathrm{m/s^2}\cdot 55.86\,\mathrm{m}} = -33.09\,\mathrm{m/s}\,.$$

(3) $y = -(x-5)(x-1)$ and so the points of intersection correspond to $x = 1$ and $x = 5$. Using $y' = -2x + 6$, we deduce that the slope is 4 at $x = 1$ and -4 at $x = 5$.

(4) (a) $V = \pi r^2 h$. The question does not make it clear whether the cylinder is closed at one, both, or none of its two ends. In the following, it is assumed that it is closed at one end, so that $S = \pi r(2h + r)$.

(b) If the volume is held fixed as r varies, then $h = V/(\pi r^2)$; substituting this in our formula for the surface area, we obtain

$$S = \frac{2V}{r} + \pi r^2\,.$$

Differentiating with respect to r, we find

$$S' = \frac{-2V}{r^2} + 2\pi r\,.$$

But $V = \pi r^2 h$ and so the instantaneous rate of change is $2\pi(r-h)$.

(c) If the surface area is held fixed, then

$$h = \frac{1}{2}\left(\frac{S}{\pi r} - r\right)\,.$$

Substituting this in our formula for the volume, we obtain

$$V = \frac{1}{2}\left(Sr - \pi r^3\right)\,.$$

Differentiating with respect to r and then expressing S in terms of r and h, we find that the instantaneous rate of change is $-\pi r(r-h)$.

(5) (a) Differentiating both sides, we find

$$1 \cdot f(x) + x f'(x) = 0 \,.$$

Hence

$$f'(x) = -\frac{f(x)}{x} = -\frac{1}{x^2} \,.$$

(b)

$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}} \,.$$

(6) Differentiate both sides of $\mathrm{e}^{f(x)} = x$ gives, by the chain rule,

$$\mathrm{e}^{f(x)} f'(x) = 1 \,.$$

Hence

$$f'(x) = \frac{1}{\mathrm{e}^{f(x)}} = \frac{1}{x} \,.$$

(7) (a) Differentiating both sides of $\sin f(x) = x$ and using the chain rule, we find, after some re-arrangement

$$f'(x) = \frac{1}{\cos f(x)} = \frac{1}{\sqrt{1 - \sin^2 f(x)}} = \frac{1}{\sqrt{1 - x^2}} \,.$$

(b) Differentiating both sides of $\cos f(x) = x$ and using the chain rule, we find, after some re-arrangement

$$f'(x) = \frac{-1}{\sin f(x)} = \frac{-1}{\sqrt{1 - \cos^2 f(x)}} = \frac{-1}{\sqrt{1 - x^2}} \,.$$

(c) Differentiating both sides of $\tan f(x) = x$ and using the chain rule, we find, after some re-arrangement

$$f'(x) = \cos^2 f(x) = \frac{\cos^2 f(x)}{\cos^2 f(x) + \sin^2 f(x)} = \frac{1}{1 + \tan^2 f(x)} = \frac{1}{1 + x^2} \,.$$

(8) By repeated use of the chain rule, we find

$$f'(x) = \mathrm{e}^{x + \mathrm{e}^x + \mathrm{e}^{\mathrm{e}^x}} \,.$$

(9) For y to be a well-defined function, we need to consider the cases $y < 0$ and $y \geq 0$ separately. For the sake of brevity, we consider the latter case only, i.e.

$$y = \sqrt{r^2 - x^2} \quad \text{where } |x| \leq r \,.$$

Implicit differentiation of the equation for the circle gives

$$2yy' + 2x = 0 \,. \text{ Hence } y' = -\frac{x}{y} = \frac{-x}{\sqrt{r^2 - x^2}} \,.$$

This is the same as the result obtained by direct differentiation of y and use of the chain rule. For the second part of the question, we start with the obvious equality

$$\frac{1}{\sqrt{1 + (y')^2}} = \frac{\sqrt{r^2 - x^2}}{r} = \frac{y}{r} \,.$$

Differentiating both sides yields

$$\frac{y'y''}{[1+(y')^2]^{3/2}} = \frac{y'}{r}$$

and the desired result follows easily.

(10) Use `Maple`'s `implicitdiff` command.