MATH11007 SOLUTION 4: APPLICATIONS OF THE DERIVATIVE

(1) (a) The equation of the parabola is

$$y = h\left(\frac{2x}{L}\right)^2 .$$

Then $y' = 8hx/L^2 = 4h/L$ at $x = L/2$. Hence the chain meets the tower at an angle $\pi/2 - \theta$, where $\tan\theta = 4h/L$. This gives an angle of 68°.

(b) We only give a brief outline. Set

$$f(a) = h - a\cosh\frac{L}{2a} + a .$$

Then Newton's method consists of the recurrence

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} .$$

This leads to $a \approx 272$. The resulting catenary is very near to the parabola used earlier, and the angle obtained is, after rounding to the nearest degree, 68°, just as before.

(2) The derivatives are $3x^2$ and $4x$ respectively, so the slopes are the same at $(0,2)$. The slopes at $(2,10)$ are $m_1 = 12$ and $m_2 = 8$ respectively. We then use the formula

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2} .$$

(3) The curves meet when $x^2 = 9$ and $y^2 = 1$. The slope of the ellipse at (x,y) is $m_1 = -4x/(9y)$, and that of the hyperbola is $m_2 = x/(4y)$. Then

$$m_1 m_2 = -\frac{x^2}{9y^2}$$

and this equals -1 at the points of intersection.

(4)

$$y' = 2\sum_{i=1}^{n}(x - a_i) .$$

This vanishes when

$$nx = \sum_{i=1}^{n} a_i .$$

Since $y'' = 2n > 0$, this value of x minimises y.

(5) The equation of the tangent line is $y = y_0 - 2x_0(x - x_0)$. It crosses the horizontal axis at

$$x = b := x_0 + \frac{y_0}{2x_0} = x_0 + \frac{a^2 - x_0^2}{2x_0}$$

and the vertical axis at

$$y = h := y_0 + 2x_0^2 = a^2 + x_0^2\,.$$

The area of the triangle is then $bh/2$. Differentiating with respect to x_0 and equating to 0, we obtain that the area is minimised when $x_0 = a/\sqrt{3}$.

(6)

(a) n^n ; (b) 1 ; (c) $(n-1)\ln 2$; (d) $\begin{cases} -1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$;

(e) 5 ; (f) 0 .

(7) The data needed to sketch the curves is presented in the following tabular form:

	$x=0$	odd/even	$\lim_{x\to-\infty}$	$\lim_{x\to\infty}$	min	max	infl	asympt
(a)	1	even	∞	∞	±1	0	–	–
(b)	0	odd	0	0	-1	1	–	–
(c)	5	–	∞	∞	2	–	0	–
(d)	0	odd	0	0	–	–	–	$y=\pm1$
(e)	–	odd	$-\infty$	∞	2	-2	–	$y=0$
(f)	`implicitplot(y*y+x*(x-2)=0,x=0..2,y=-1..1);`							
(g)	`implicitplot(y*y=x*x*x,x=0..2,y=-1..1);`							
(h)	`implicitplot(y*y*y=x*x,x=-1..1,y=0..2);`							
(i)	–	–	$-\infty$	∞	–	–	–	$y=\pm1$
(j)	$\sin 6$	even	0	0	$\pm\sqrt{\frac{4}{\pi}-1}$	$0,\ \pm\sqrt{\frac{12}{\pi}-1}$	–	–
(k)	`plot(sqrt(1-x)+sqrt(x+2),x=-2..1);`							
(l)	`implicitplot(abs(y)^a +abs(x)^a=1,x=-1..1,y=-1..1);`							

Here, "min" means "local minimum" etc. Note that, before you can use the `Maple` command `implicitplot`, you must issue the command

```
> with(plots);
```

(8) (a) Set $f(x) = x^p - a$. Then

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)} = -\frac{x_n^p - a}{p\,x_n^{p-1}}\,.$$

(b) Set $f(x) = \mathrm{e}^x - p$. Then

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)} = -\frac{\mathrm{e}^{x_n} - p}{\mathrm{e}^{x_n}}\,.$$

(9) If f is even, then, for every $x \in \mathbb{R}$, $f(-x) = f(x)$. Differentiate both sides with respect to x. Using the chain rule for the left-hand side, we obtain

$$-f'(-x) = f'(x)$$

and so f' is odd. Using similar arguments, it is straightforward to show also that the derivative of an odd function is even.

(10) The perimeter is $x+1-\lambda x+2y$. Using $xy = 1$, we deduce that the function to minimise is

$$p(x) = x + 1 - \lambda x + 2/x\,.$$

Differentiating and equating to zero, we find

$$x = \sqrt{\frac{2}{1-\lambda}} \quad \text{and} \quad y = \sqrt{\frac{1-\lambda}{2}}\,.$$