CONTOUR INTEGRATION IN THE PRESENCE OF BRANCH CUTS

Abstract. The use of the Bromwich inversion formula in the presence of branch cuts is illustrated in the context of a particular example.

1. Motivation

Let $x > 0$. We ask for the function $f = f(t)$ with the Laplace transform

$$L_f(p) = \frac{\mathrm{e}^{-x\sqrt{p}}}{p}\,.$$

The Bromwich inversion formula expresses f as an integral in the complex p-plane:

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \mathrm{e}^{-x\sqrt{p}+pt}\,\frac{\mathrm{d}p}{p} \quad \text{for } t > 0\,. \tag{1.1}$$

The integrand— viewed as a function of p— has a singularity at $p = 0$. There are two reasons for this: firstly, p appears as a denominator; secondly $\sqrt{p}$ appears in the exponent. The Residue Theorem makes clear how poles contribute in the calculation of contour integrals, but in the present case, $p = 0$ is *not* a pole; the appearance of $\sqrt{p}$ means that it is a *branch point* of the integrand. In what follows, we discuss the basic facts concerning branch points and explain how to evaluate contour integrals in the presence of such singularities.

2. What is a branch point?

Recall that a function $w(z)$ is *analytic* at z_0 if it can be expanded in a Taylor series at $z = z_0$ and the series has a strictly positive radius of convergence. Thus, for example,

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{z_0}}{n!}(z - z_0)^n$$

and this shows that the exponential function is analytic at *every* point of the complex plane. Recall also that a function $w(z)$ that is not analytic at z_0 has a *pole* there if there is a positive natural number n such that $(z - z_0)^n w(z)$ is analytic at z_0; the smallest such number n is then called the *order* of the pole. Thus, for example,

$$\frac{\sin z}{z^2}$$

has a pole of order 1 (i.e. a simple pole) at $z = 0$.

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2.1. **The square root function.** We would now like to define the square root $w(z)$ of a complex number z. For numbers $z \geq 0$, we define it as the non-negative solution $w = w(z)$ of

$$w^2 = z\,.$$

Then for complex z, we remark that $z \neq 0$ may be expressed in a unique way as

$$z = |z|\mathrm{e}^{\mathrm{i}\theta}\,, \quad \theta \in (-\pi,\, \pi]\,.$$

Therefore, we set

$$w(z) := \sqrt{|z|}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\,. \tag{2.1}$$

The function thus defined has the following features:

(1) For $z_0 \notin (-\infty,\, 0]$, $w(z)$ is analytic at z_0 and

$$w'(z_0) = \frac{1}{2w(z_0)}\,.$$

(2) For $R > 0$,

$$\lim_{0<\varepsilon\to 0} w(-R + \mathrm{i}\varepsilon) = \mathrm{i}\sqrt{R} \;\text{ and }\; \lim_{0<\varepsilon\to 0} w(-R - \mathrm{i}\varepsilon) = -\mathrm{i}\sqrt{R}\,.$$

In words: if we evaluate the square root of a complex number that is just above the negative part of the real line, we get a value with a positive imaginary part; on the other hand, if the number is just below the negative part of the real line, then its square root has a negative imaginary part.

(3) Our square root function is therefore *discontinuous* at every point $z_0 \leq 0$.

(4) Let $z_0 \leq 0$. Then, w is not analytic at z_0 (the derivative $w'(z_0)$ does not exist); furthermore, there is no natural number n such that $(z - z_0)^n w(z)$ is analytic at z_0. So z_0 is *not* a pole.

The half-line $z < 0$ is called a *branch cut* of the square root function; the points 0 and ∞ at the ends of the branch cut are called *branch points.*

2.2. **Other functions with branch cuts.** Branch cuts need to be introduced for other functions of a complex variable. The most common examples are

$$\ln z\,, \quad z^\alpha \;\text{ with } \alpha \in (0,1)$$

and other functions constructed from them. The object of introducing branch cuts is to make the function single-valued.

3. Contours and branch cuts

Let us now return to the calculation of the integral (1.1). If we wish to exploit the Residue Theorem, we must "close" the contour without crossing the branch cut. We choose a large positive number R and use the contour

$$C := \cup_{j=1}^{6} C_j$$

where

(1) C_1 is the vertical line segment going from $\gamma - \mathrm{i}R$ to $\gamma + \mathrm{i}R$. We use the parametrisation

$$p = \gamma + \mathrm{i}s\,, \quad s \in [-R, R]\,.$$

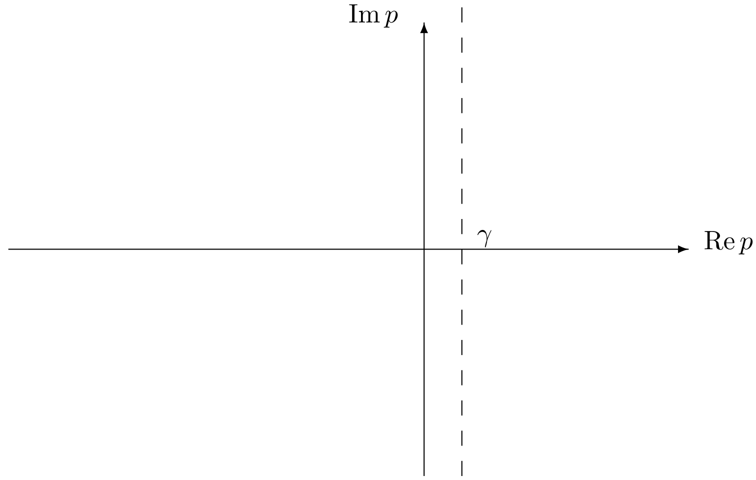

FIGURE 1. Draw the contour C.

(2) C_2 is the arc of the circle of radius R centered at γ going anticlockwise from $\gamma + \mathrm{i}R$ to $\gamma - R$. We use the parametrisation

$$p = \gamma + R\mathrm{e}^{\mathrm{i}\theta}\,, \quad \theta \in [\pi/2, \pi)\,.$$

(3) This takes us to a point just above the branch cut. Since we must not cross the cut, C_3 follows the cut from $\gamma - R$ to $-\varepsilon$ where ε is a positive number destined to tend to zero. We use the parametrisation

$$p = s + \mathrm{i}0+\,, \quad s \in [\gamma - R, -\varepsilon]\,.$$

(4) Next, the contour winds around the origin in order to get to the other side of the cut: C_4 is the arc of the circle of radius ε centered at the origin going clockwise from $-\varepsilon + \mathrm{i}0+$ to $-\varepsilon + \mathrm{i}0-$. We use the parametrisation

$$p = \varepsilon\mathrm{e}^{\mathrm{i}\theta}\,, \quad \theta \in (\pi, -\pi)\,.$$

(5) We have now reached a point just below the branch cut. C_5 follows the cut from $-\varepsilon$ to $\gamma - R$. We use the parametrisation

$$p = s + \mathrm{i}0-\,, \quad s \in [-\varepsilon, \gamma - R]\,.$$

(6) Finally, C_6 is the arc of the circle of radius R centered at γ that goes anticlockwise from $\gamma - R$ to $\gamma - \mathrm{i}R$. We use the parametrisation

$$p = \gamma + R\mathrm{e}^{\mathrm{i}\theta}\,, \quad \theta \in (-\pi, -\pi/2]\,.$$

You are encouraged to draw the contour in Figure 3. A shifted version of the contour appears in Figure 4 of the solution sheet G (but the picture is not very good!).

Since

$$\frac{1}{2\pi\mathrm{i}} L_f(p)\,\mathrm{e}^{pt} := \frac{\mathrm{e}^{-x\sqrt{p}+pt}}{2\pi\mathrm{i}p}$$

is analytic *inside* the contour C, the Residue Theorem says

$$\frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{e}^{-x\sqrt{p}+pt}}{p}\,\mathrm{d}p = 0\,.$$

and hence, by the Bromwich inversion formula

$$f(t) = -\lim_{R\to\infty,\ \varepsilon\to 0} \sum_{j=2}^{6} \frac{1}{2\pi \mathrm{i}} \int_{C_j} \frac{\mathrm{e}^{-x\sqrt{p}+pt}}{p}\, \mathrm{d}p\,. \tag{3.1}$$

We now discuss the five limits on the right-hand side.

Since $\cos\theta < 0$ along the arcs C_2 and C_6, the reasoning used in the lectures leads to

$$\lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{C_2} \frac{\mathrm{e}^{-x\sqrt{p}+pt}}{p}\, \mathrm{d}p = \lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{C_6} \frac{\mathrm{e}^{-x\sqrt{p}+pt}}{p}\, \mathrm{d}p = 0\,.$$

Also

$$\frac{1}{2\pi \mathrm{i}} \int_{C_4} \frac{\mathrm{e}^{-x\sqrt{p}+pt}}{p}\, \mathrm{d}p = \frac{1}{2\pi} \int_{\pi}^{-\pi} \exp\left[-x\sqrt{\varepsilon}\,\mathrm{e}^{\mathrm{i}\theta/2} + \varepsilon\,\mathrm{e}^{\mathrm{i}\theta} t\right] \mathrm{d}\theta \xrightarrow{\varepsilon\to 0} -1\,.$$

For C_3, we have

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}} \int_{C_3} \exp\left[-x\sqrt{p}+pt\right] \frac{\mathrm{d}p}{p} &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma-R}^{-\varepsilon} \exp\left[-\mathrm{i}x\sqrt{-s}+st\right] \frac{\mathrm{d}s}{s} \\ &= -\frac{1}{2\pi \mathrm{i}} \int_{\varepsilon}^{R-\gamma} \exp\left[-\mathrm{i}x\sqrt{s}-st\right] \frac{\mathrm{d}s}{s}\,.\end{aligned}$$

The point here is that C_3 is just above the cut, so that $\sqrt{p} = \mathrm{i}\sqrt{-p}$. When the integral is along C_5 instead, we are just below the cut and so $\sqrt{p} = -\mathrm{i}\sqrt{-p}$. We therefore obtain

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}} \int_{C_3\cup C_5} \exp\left[-x\sqrt{p}+pt\right] \frac{\mathrm{d}p}{p} &== \frac{1}{2\pi \mathrm{i}} \int_{\varepsilon}^{R-\gamma} \mathrm{e}^{-st} \left[\mathrm{e}^{\mathrm{i}x\sqrt{s}} - \mathrm{e}^{-\mathrm{i}x\sqrt{s}}\right] \frac{\mathrm{d}s}{s} \\ \frac{1}{\pi} \int_{\varepsilon}^{R-\gamma} \mathrm{e}^{-st} \sin\left(x\sqrt{s}\right) \frac{\mathrm{d}s}{s} &\xrightarrow[\varepsilon\to 0]{R\to\infty} \frac{1}{\pi} \int_{0}^{\infty} \mathrm{e}^{-st} \sin\left(x\sqrt{s}\right) \frac{\mathrm{d}s}{s}\,.\end{aligned}$$

When we put all the foregoing results into Equation (**??**), we find

$$f(t) = 1 - \frac{1}{\pi} \int_{0}^{\infty} \mathrm{e}^{-st} \sin\left(x\sqrt{s}\right) \frac{\mathrm{d}s}{s}\,.$$