PRODUCTS OF RANDOM MATRICES

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Abstract. This is a light introduction to the theory of products of random matrices, confined to the $2 \times 2$ case. After some motivation, we discuss and illustrate Furstenberg’s theorem.

1. Motivation

Products of random matrices arise naturally in the study of wave propagation through a layered, disordered medium. The following example is drawn from quantum mechanics [2], but the same mathematical features are also found in acoustic or electromagnetic contexts.

The quantum mechanical description of an electron propagating through a one-dimensional semi-infinite crystal uses the Schrödinger equation for the wave function, say $\psi$, of the electron:

$$-\psi''(x) + V(x)\psi(x) = E\psi(x), \quad x > 0.$$  

(1.1)

In this model, $x$ denotes the position along the crystal, the “potential” $V$ determines the interaction between the crystal’s atoms and the electron, and $E$ is the electron’s energy. The values of $E$ such that the wave function is square integrable are of particular interest, as they correspond to bound states of the electron.

Next, we consider the particular case where

$$V = \sum_{n=1}^{\infty} m_n\delta_{x_n}.$$  

(1.2)

Here, $x_n$ is the general term of a positive, increasing sequence, $m_n$ is the general term of a real sequence, and $\delta$ is the Dirac delta, i.e. for every nice function $f$,

$$\int_{\mathbb{R}} f(x)\delta_{x_n} := f(x_n).$$

One interpretation for this choice of potential is as follows: there are impurities in the crystal located at $x_1, x_2, \ldots$, and $m_n$ is the “strength” of the impurity at $x_n$; the electron moves freely between two consecutive impurities.

With this choice, the solution of the equation (1.1) is discontinuous at the $x_n$, but may be obtained in a piecewise fashion: For $x_n < x < x_{n+1}$ and positive $E = k^2$,

$$\begin{pmatrix} \psi'(x) \\ k\psi(x) \end{pmatrix} = \begin{pmatrix} \cos[k(x-x_n)] & -\sin[k(x-x_n)] \\ \sin[k(x-x_n)] & \cos[k(x-x_n)] \end{pmatrix} \begin{pmatrix} 1 & m_n/k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi'(x_n^-) \\ k\psi(x_n^-) \end{pmatrix}.$$
Iterating, we deduce
\[
\begin{pmatrix}
\psi'(x_{n+1}) \\
\kappa \psi(x_{n+1})
\end{pmatrix} = \prod_{j=1}^{n} A_j \begin{pmatrix}
\psi'(0) \cos(k\ell_1) - k\psi(0) \sin(k\ell_1) \\
\psi'(0) \sin(k\ell_1) + k\psi(0) \cos(k\ell_1)
\end{pmatrix}
\]
where
\[
A_n := \begin{pmatrix}
\cos(k\ell_{n+1}) & -\sin(k\ell_{n+1}) \\
\sin(k\ell_{n+1}) & \cos(k\ell_{n+1})
\end{pmatrix} \begin{pmatrix}
1 & \frac{m_n}{n} \\
0 & 1
\end{pmatrix}.
\]
and \(\ell_n = x_n - x_{n-1}\), with \(x_0 = 0\).

The study of the behaviour of the solution for large \(x\) is thus reduced to the study of the behaviour of the product
\[
\prod_{j=1}^{n} A_j
\]
for large \(n\). Suppose first that the impurities are uniformly spaced, and of uniform strength; denote the spacing by \(\ell\) and the strength by \(h\). Then the product is simply \(A^n\), where
\[
A := \begin{pmatrix}
\cos(k\ell) & -\sin(k\ell) \\
\sin(k\ell) & \cos(k\ell)
\end{pmatrix} \begin{pmatrix}
1 & \frac{m}{n} \\
0 & 1
\end{pmatrix}.
\]
The asymptotic behaviour of the product is then readily determined by examining the eigenvalues and eigenvectors of \(A\).

A useful model of a disordered crystal is obtained if we assume the strength of the impurities (the \(m_n\)) and the spacing between them (the \(\ell_n\)) to be random variables. Hence we are led to the theme of these lectures.

2. The Lyapunov Exponent

We proceed to study the large-\(n\) behaviour of the product (1.5) when the \(A_n\) are random. More precisely, we shall be concerned with the limit
\[
\frac{1}{n} \ln |\prod_{j=1}^{n} A_j x| \quad \text{as} \quad n \to \infty
\]
where \(x\) is some non-zero vector and \(|\cdot|\) is the familiar euclidean norm. Also, we shall restrict our attention to the case where the \(A_n\) are independent and identically distributed. We denote by \(\mu\) the distribution from which they are drawn, i.e. for every set \(M\) of matrices,
\[
P(A_n \in M) = \int_M \mu(dA).
\]
To get a feel for the question, consider the case where the \(A_n\) are numbers (1 \times 1 matrices!). Then \(|\cdot|\) is the absolute value and we have
\[
\frac{1}{n} \ln |\prod_{j=1}^{n} A_j x| = \frac{1}{n} \ln \left( \prod_{j=1}^{n} |A_j| \right) + \frac{\ln |x|}{n} = \frac{1}{n} \sum_{j=1}^{n} \ln |A_j| + \frac{\ln |x|}{n}
\]
\[
\sim \frac{1}{n} \sum_{j=1}^{n} \ln |A_j| \xrightarrow{a.s.} n \to \infty \mathbb{E}(\ln |A|)
\]

\[
\frac{1}{n} \ln |\prod_{j=1}^{n} A_j| = \frac{1}{n} \ln \left( \prod_{j=1}^{n} |A_j| \right) + \frac{\ln |x|}{n} = \frac{1}{n} \sum_{j=1}^{n} \ln |A_j| + \frac{\ln |x|}{n}
\]
\[
\sim \frac{1}{n} \sum_{j=1}^{n} \ln |A_j| \xrightarrow{a.s.} n \to \infty \mathbb{E}(\ln |A|)
\]
where $A$ is $\mu$-distributed. This result is nothing but the familiar Law of Large Numbers, which says that, if one draws numbers repeatedly and independently from the same distribution, then the average value after many draws approaches the mean of that distribution. In the $1 \times 1$ case, the growth of the product is therefore easily expressed in terms of the distribution $\mu$, since

$$E(\ln |A|) = \int_{\text{supp}(\mu)} \ln |A| \mu(dA).$$

Next, consider the $d \times d$ case, $d > 1$. Without loss of generality, we may suppose that the $A_n$ have determinant $\pm 1$, i.e. $A_n \in \text{SL}(d, \mathbb{R})$. Denote by $|A|$ the norm of the matrix $A$, i.e.

$$|A| := \sup_{|x| \leq 1} |Ax|.$$

For typical $A, B \in \text{SL}(d, \mathbb{R})$, we have

$$|AB| \leq |A||B|$$

rather than the strict equality of the $d = 1$ case. So the foregoing argument breaks down. Nevertheless, as we shall see, the result

$$\frac{1}{n} \ln \left| \prod_{j=1}^{n} A_j x \right| \xrightarrow{a.s.} \gamma$$

still holds. The number $\gamma$ is called the Lyapunov exponent of the product, but the formula for it takes a more complicated form than in the $d = 1$ case:

$$\gamma = \int_{P(\mathbb{R}^d)} \int_{\text{SL}(d, \mathbb{R})} \ln \frac{|Ax|}{|x|} \mu(dA)\nu(dx)$$

where $\nu$ is a certain measure on the projective space $P(\mathbb{R}^d)$.

3. **The projective space**

We say that two vectors $x$ and $y$ in $\mathbb{R}^d$ have the same direction if one is a scalar multiple of the other, i.e. there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$y = \lambda x.$$

This defines an equivalence relation in $\mathbb{R}^d$; the set of all directions can be partitioned into equivalence classes, and each equivalence class can be identified with a straight line through the origin. The set of all such lines is called the projective space and is denoted $P(\mathbb{R}^d)$.

Let us elaborate the particular case $d = 2$; we then speak of the projective line. By definition, the elements of the projective line are the straight lines in $\mathbb{R}^2$ that pass through the origin. To specify a particular member of the projective line, we may use (the reciprocal of) its slope. The reciprocal of this slope, call it $\overline{\tau}$, may be finite or infinite. Thus, we have a bijection between the projective line and the set of “numbers” $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. With some abuse of notation, we shall sometimes write

$$P(\mathbb{R}^2) = \overline{\mathbb{R}}.$$

This is how one should read the right-hand side of Equation (2.2): the number

$$\frac{|Ax|}{|x|} = \left| \frac{A}{x} \right|$$
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depends only on \( A \) and the direction of \( x \) and so, in the case \( d = 2 \), we write

\[
\frac{|Ax|}{|x|} = \frac{|A\begin{pmatrix} x \\ 1 \end{pmatrix}|}{|\begin{pmatrix} x \\ 1 \end{pmatrix}|}.
\]

The relevance of the projective space to our problem may be understood intuitively as follows. Let \( d = 2 \) and write the product in the column form

\[
\prod_{j=1}^{n} A_j = (p_n, q_n).
\]

Recall the geometrical interpretation of the determinant in the \( 2 \times 2 \) case: its modulus is the area of the parallelogram spanned by the columns. We see that unimodularity implies that

\[
|p_n| |q_n| \sin \theta_n = 1
\]

where \( \theta_n \) is the angle between the columns. Hence, if we show that the columns tend to align along the same direction, then \( \theta_n \to 0 \) and at least one of \( |p_n| \) or \( |q_n| \) must grow.

Let

\[
A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}).
\]

We are interested in quantifying how the direction of a vector changes when it is multiplied by \( A \). Let

\[
x = \begin{pmatrix} x \\ y \end{pmatrix} \neq \mathbf{0}.
\]

Since, by assumption, the determinant of \( A \) has unit modulus, the vector \( Ax \) is non-zero and so lies along a line in \( \mathbb{R}^2 \). The reciprocal of the slope of this line is

\[
\frac{ax + by}{cx + dy} = \frac{ax/y + b}{cx/y + d}.
\]

The matrix \( A \) has “acted” on the line of slope equal to the reciprocal of \( \overline{\tau} = x/y \) and mapped it to another line with slope whose reciprocal is

\[
\mathcal{T}_A(\overline{\tau}) = \begin{cases} 
\frac{a}{c} & \text{if } \overline{\tau} = \infty \\
\frac{a \overline{\tau} + b}{c \overline{\tau} + d} & \text{otherwise}.
\end{cases}
\]

We shall henceforth drop the awkward bar and write \( x \) instead of \( \overline{\tau} \). The action \( \mathcal{T}_A : \mathbb{R} \to \mathbb{R} \) of the matrix \( A \) on the projective line will sometimes be denoted by

\[
\mathcal{T}_A(x) = A \cdot x.
\]

4. THE STATIONARY MEASURE

Now let \( \nu \) be a distribution on the projective line, and suppose that \( X \) is a \( \nu \)-distributed random variable. Writing \( \mathcal{T}_j \) for the action of \( A_j \) on the projective line, it is easily seen that

\[
\prod_{j=1}^{n} A_j \cdot X = \mathcal{T}_n \circ \cdots \circ \mathcal{T}_1(X) =: Y_n(X).
\]
The sequence \( \{Y_n\}_{n \in \mathbb{N}} \) is a random process that is a Markov chain on the projective line.

**Definition 4.1.** Let \( \mu \) be a probability distribution on \( \text{SL}(2, \mathbb{R}) \). We say that \( \nu \) is stationary for \( \mu \) if \( A \cdot X \) is also \( \nu \)-distributed.

The \( \mu \)-stationary probability measure \( \nu \) (if it exists and is unique) is the measure that appears in Equation (2.2); it is a very important tool in the study of the statistics of the Markov chain \( \{Y_n\}_{n \in \mathbb{N}} \). Indeed, suppose that \( X \) is \( \nu \)-distributed. Then every \( Y_n \) in the above Markov chain is also \( \nu \)-distributed and, although the \( Y_n \) are not independent, a kind of law of large numbers—the ergodic theorem—does hold: for every \( \nu \)-measurable function \( f: \mathbb{P}(\mathbb{R}^2) \to \mathbb{R} \),

\[
\frac{1}{n} \sum_{j=1}^{n} f(Y_n) \xrightarrow{n \to \infty} \mathbb{E}(f(X)) = \int_{\mathbb{P}(\mathbb{R}^2)} f(x) \nu(dx).
\]

How can one find \( \nu \)? By definition of \( \mu \)-stationarity, for every \( \nu \)-measurable function \( f: \mathbb{P}(\mathbb{R}^2) \to \mathbb{R} \), we must have

\[
\int_{\mathbb{P}(\mathbb{R}^2)} f(x) \nu(dx) = \int_{\mathbb{P}(\mathbb{R}^2)} \int_{\text{SL}(2, \mathbb{R})} f(A \cdot x) \mu(dA) \nu(dx).
\]

If we suppose that \( \nu \) has a density, one can deduce an integral equation for it.

The existence and uniqueness of a measure stationary for \( \mu \) is intimately connected with the convergence of a certain continued fraction. Let the \( A_n \) be \( \mu \)-distributed, and write \( \Delta_n = a_n d_n - b_n c_n \). Then

\[
(4.3) \quad \mathcal{T}_n(x) = a_n/c_n - \frac{\Delta_n/c_n^2}{d_n/c_n + x}.
\]

Set

\[
(4.4) \quad X_n(x) := \mathcal{T}_1 \circ \cdots \circ \mathcal{T}_n(x).
\]

This \( X_n \) resembles the \( Y_n \) in Equation (4.1), but the order in which the linear fractional transformations are applied is reversed. Suppose that the sequence \( \{X_n\}_{n \in \mathbb{N}} \) converges to a limit, say \( X \), independent of \( x \). We express this as

\[
(4.5) \quad X := a_1/c_1 - \frac{\Delta_1/c_1^2}{d_1/c_1 + a_2/c_2 - \frac{\Delta_2/c_2^2}{d_2/c_2 + a_3/c_3 - \frac{\Delta_3/c_3^2}{d_3/c_3 + \cdots}}},
\]

If \( A \) is \( \mu \)-distributed, then

\[
A \cdot X =: \mathcal{T}_A(X) = a/c - \frac{\Delta/c^2}{d/c + X}
\]

has the same distribution as \( X \). Hence the distribution of \( X \) is stationary for \( \mu \). The fact that \( X \) is independent of \( x \) implies that there can be no other \( \mu \)-stationary measure.
5. Furstenberg’s Theorem

Furstenberg [3] and others developed a theory with the aim of generalising the law of large numbers to a wide class of groups. Here we apply it to the case where the group is \( G := \text{SL}(2, \mathbb{R}) \). Using the matrix norm introduced in §2, we can speak of limits of sequences, and of bounded and closed sets in \( G \). Given a probability measure \( \mu \) on \( G \), we denote by \( G_\mu \) the smallest closed subgroup of \( G \) containing the support of \( \mu \).

**Definition 5.1.** A subgroup \( \tilde{G} \) of \( G \) is said to be **strongly irreducible** if there is no finite union

\[
L = \bigcup_{j=1}^{m} L_j
\]

of one-dimensional subspaces of \( \mathbb{R}^2 \) such that

\[
A(L) = L
\]

for some \( A \in \tilde{G} \).

By extension, we say that the measure \( \mu \) is strongly irreducible if \( G_\mu \) is strongly irreducible.

The following criterion will be useful.

**Proposition 5.1.** If \( G_\mu \) is unbounded, then \( G_\mu \) is strongly irreducible if and only if the following holds: For every \( x \in P(\mathbb{R}^2) \), the set

\[
\{ A \cdot x : A \in G_\mu \}
\]

contains more than two elements.

**Theorem 1.** Suppose that

\[
\mathbb{E}(\ln |A|) < \infty.
\]

Suppose also that \( G_\mu \) is strongly irreducible and unbounded. Then the following statements are true:

1. There exist a unique measure \( \nu \) on \( P(\mathbb{R}^2) \) that is stationary for \( \mu \).
2. For every non-zero vector \( x \), we have

\[
\frac{1}{n} \ln \left| \prod_{j=1}^{n} A_j x \right| \xrightarrow{a.s.} \gamma \quad \text{as } n \to \infty
\]

where

\[
\gamma = \int_{\mathbb{R}} \int_{\text{SL}(2, \mathbb{R})} \ln \left| A \begin{pmatrix} x \\ 1 \end{pmatrix} \right| \mu(dA) \nu(dx).
\]

3. \( \gamma > 0 \).

6. Some simple examples

Let us illustrate the foregoing discussion with some simple examples.
6.1. The Fibonacci sequence. Our first example is deterministic! The Fibonacci sequence \( \{u_n\}_{n \in \mathbb{N}} \) is defined by the recurrence relation
\[(6.1) u_{n+1} = u_n + u_{n-1}, \quad n \in \mathbb{N},\]
with \( u_0 = u_1 = 1 \). We can write this in the form
\[
\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_1 \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A_1^n \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]
where \( A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \).
We may think of \( A_1^n \) as a product of “random” matrices with a distribution \( \mu \) whose mass is concentrated at \( A_1 \). We write
\[ \mu = \delta_{A_1}. \]
It is straightforward to check that the corresponding \( G_\mu \) is unbounded. However, \( \mu \) is not irreducible. (Consider the direction of an eigenvector of \( A_1 \)). So the hypothesis of Furstenberg’s theorem is not verified. Nevertheless, we shall see that the conclusion of the theorem holds, thus showing that the hypothesis is sufficient but not necessary.

Equation (4.5) gives
\[(6.2) X = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}} = 1 + \frac{\sqrt{5}}{2}.
\]
The distribution of this “random” variable is thus
\[ \nu = \delta_X \]
and so, using the fact that \( X^2 = X + 1 \), the formula (2.2) yields
\[
\gamma = \int_{\mathbb{R}} \int_{\text{SL}(2, \mathbb{R})} \ln \left| A \begin{pmatrix} x \\ 1 \end{pmatrix} \right| \delta_{A_1} \delta_X = \int_{\mathbb{R}} \ln \left| A_1 \begin{pmatrix} x \\ 1 \end{pmatrix} \right| \delta_X
\]
\[
= \ln \left| A_1 \begin{pmatrix} X \\ 1 \end{pmatrix} \right| = \ln \left| \begin{pmatrix} 1 + X \\ X \end{pmatrix} \right| = \ln \left| \begin{pmatrix} X^2 \\ X \end{pmatrix} \right| = \ln X > 0.
\]

6.2. A random Fibonacci sequence. Consider the following randomised version of the previous example:
\[(6.3) u_{n+1} = a_n u_n + u_{n-1}, \quad n \in \mathbb{N},\]
with \( u_0 = u_1 = 1 \), where the \( a_n \) are independent random variables taking the values \( \pm 1 \) with equal probability. Hence
\[
\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \prod_{j=1}^{n} A_j \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]
where the $A_n$ are drawn from

$$\left\{ \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

and each element has equal probability.

$G_n$ is larger than in the previous (deterministic) example, and so it is immediate that Furstenberg’s theorem holds.

The following illustrates the kind of manipulations involved in the calculation of the Lyapunov exponent:

$$\gamma = \int_{\mathbb{R}} \int_{G_n} \ln \left| A \begin{pmatrix} x \\ 1 \end{pmatrix} \right| \mu(dA) \nu(dx) = \frac{1}{2} \int_{\mathbb{R}} \int_{G_n} \ln \left| A \begin{pmatrix} x \\ 1 \end{pmatrix} \right|^2 \nu(dx)$$

$$= \frac{1}{2} \int_{\mathbb{R}} \int_{G_n} \ln \left( ax + 1 \right)^2 + x^2 \mu(dA) \nu(dx)$$

$$= \frac{1}{2} \int_{\mathbb{R}} \int_{G_n} \ln \left( \frac{x^2}{1 + x^2} \right) \left[ 1 + (A \cdot x)^2 \right] \mu(dA) \nu(dx)$$

$$= \frac{1}{2} \int_{\mathbb{R}} \int_{G_n} \ln \left( \frac{x^2}{1 + x^2} \right) \mu(dA) \nu(dx) + \frac{1}{2} \int_{\mathbb{R}} \int_{G_n} \ln \left[ 1 + (A \cdot x)^2 \right] \mu(dA) \nu(dx).$$

At this point, we observe that, in the first of these integrals, the integrand is independent of $A$. Hence

$$\frac{1}{2} \int_{\mathbb{R}} \int_{G_n} \ln \left( \frac{x^2}{1 + x^2} \right) \mu(dA) \nu(dx) = \frac{1}{2} \int_{\mathbb{R}} \ln \frac{x^2}{1 + x^2} \nu(dx).$$

Furthermore, using the fact that $\nu$ is stationary for $\mu$ (cf. Equation 4.2),

$$\frac{1}{2} \int_{\mathbb{R}} \int_{G_n} \ln \left[ 1 + (A \cdot x)^2 \right] \mu(dA) \nu(dx) = \frac{1}{2} \int_{\mathbb{R}} \ln \left[ 1 + x^2 \right] \nu(dx).$$

Putting these results together, we obtain a much deflated formula for the Lyapunov exponent:

$$\gamma = \int_{\mathbb{R}} \ln |x| \nu(dx).$$

Viswanath [5] considered and solved the problem of finding the $\mu$-stationary measure $\nu$ for this example. It turns that $\nu$ is not a smooth measure; the technical word for the class to which it belongs is singular continuous. There is no explicit formula for it, but the measure of any real interval may be computed exactly to any desired accuracy by means of a recursion. Then

$$\gamma \in (0.1239755980, 0.1239755995).$$

Figure 1 shows a histogram of the first 40000 terms of sequence $\{Y_n\}_{n \in \mathbb{N}}$ defined by Equation (4.1) with $x = 1$. The sequence is ergodic and so the histogram provides an “approximation” of the graph of $\nu(dx)/dx$. 
6.3. A product that does not grow. The following example is taken from [1]. Let \( \alpha > 0 \) and set
\[
D := \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .
\]

Consider the following distribution:
\[
\mu := p\delta_D + (1 - p)\delta_R , \quad p \in [0, 1] .
\]
The matrix \( R \) is a rotation matrix, i.e. \( Rx \) is the vector obtained after rotating \( x \) by the angle \( \pi/2 \). It is therefore obvious that, if \( p = 0 \), then \( \gamma = 0 \). On the other hand,
\[
D \cdot x = \alpha^2 x .
\]
Therefore, if \( p = 1 \), the columns of the corresponding product of matrices will align along the vertical or horizontal direction, depending on whether or not \( \alpha \) exceeds unity. As discussed earlier, this implies \( \gamma > 0 \). What happens if \( 0 < p < 1 \)?

Figure 2 shows a plot of the quantity
\[
\frac{1}{n} \ln \left| \prod_{j=1}^{n} A_j \right|
\]
against \( n \) in the case where \( p = 1/2 \). The suggestion is that \( \gamma = 0 \). Turning to the statement of Furstenberg’s theorem, we note that
\[
\mathbb{E}(|A_1|) = p \ln \alpha .
\]
For $0 < p < 1$, $G_\mu$ contains $D$, $R$, $D^{-1}$, $R^{-1}$ and every product of these. A little calculation shows that

$$G_\mu = \left\{ \begin{pmatrix} \beta^n & 0 \\ 0 & \beta^{-n} \end{pmatrix}, \begin{pmatrix} 0 & \beta^n \\ -\beta^{-n} & 0 \end{pmatrix} : \beta \in \{\pm \alpha, \pm 1/\alpha\}, n \in \mathbb{Z} \right\}.$$ 

In particular, for every natural number $n$, $D^n \in G_\mu$ and so $G_\mu$ is unbounded.

There only remains to examine the strong irreducibility assumption. Let $x = 0$. Then

$$\begin{pmatrix} \beta^n & 0 \\ 0 & \beta^{-n} \end{pmatrix} \cdot x = 0 \text{ and } \begin{pmatrix} 0 & \beta^n \\ -\beta^{-n} & 0 \end{pmatrix} \cdot x = \infty.$$ 

We see that $\mu$ fails to satisfy the strong irreducibility criterion contained in Proposition 5.1.

7. The Frisch–Lloyd model

We return to the problem described in §1, for which the matrices in the product are given by

$$A_n := \begin{pmatrix} \cos(k\ell_n) & -\sin(k\ell_n) \\ \sin(k\ell_n) & \cos(k\ell_n) \end{pmatrix} \begin{pmatrix} 1 & \frac{m}{k} \\ 0 & 1 \end{pmatrix}.$$ 

The corresponding linear fractional transformation is thus

$$T_A(x) = \cot(k\ell) - \frac{\csc^2(k\ell)}{m/k + \cot(k\ell) + x}.$$
We consider the case where the \( \ell_n \) are independent with a common exponential distribution of parameter \( p \), i.e. for every measurable subset \( S \subseteq \mathbb{R}_+ \),

\[
P(\ell_n \in S) = \int_S p e^{-px} \, dx.
\]

Next, we suppose that the \( m_n \) are independent random variables, independent also of the \( \ell_n \), with a common distribution whose density we shall denote by \( \varrho_h : \mathbb{R} \to \mathbb{R} \).

In §4, we stated that the corresponding \( \mu \)-stationary measure could be found by solving a certain integral equation for its density. In this particular case, however, there is an alternative.

One can work with the differential equation

\[
Z' = Z^2 + k^2 - V
\]

satisfied by the Riccati variable

\[
Z = -\frac{\psi'}{\psi}.
\]

Since \( V \) is random, \( \{Z(x) : x \in \mathbb{R}_+\} \) is a random process, and the particular form of \( V \) implies that this process has the Markov property. One may think of the process \( \{Z(x)\}_{x \geq 0} \) as a “continuous version”, up to a constant factor, of the Markov chain \( \{Y_n\}_{n \in \mathbb{N}} \) discussed in §4, in which \( x \) plays the rôle of \( n \). In particular, there is a stationary distribution for this Markov process and, according to Kotani [4], its density, say \( T \), satisfies the integro-differential equation

\[
\frac{d}{dz} [(z^2 + k^2) T(z)] = p \int_{-\infty}^{\infty} [T(z + t) - T(z)] \varrho_h(t) \, dt.
\]

The presence of the integral term makes such equations difficult to solve explicitly; see [2] for a numerical approach.

**References**


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