Homework 2

(1) An element \( v \) of a lattice \( \Lambda \subset \mathbb{R}^d \) is called \textit{primitive} if \( v \neq n \cdot w \) for all integers \( n \geq 2 \) and \( w \in \Lambda \). Show that there exists universal \( \epsilon_0 > 0 \) such that every \( \Lambda \in \mathcal{L}^1_2 \) contains at most one (up to sign) primitive \( v \) with \( \|v\| \leq \epsilon_0 \).

(2) (exponential mixing) Let \( D_2 : S^1 \to S^1 : z \mapsto z^2 \) be the doubling map on the circle. Prove that there exists \( \theta \in (0, 1) \) such that for every continuously differentiable functions \( \phi, \psi \in C^1(S^1) \),

\[
\left| \int_{S^1} \phi(D_2^n z) \psi(z) dz - \left( \int_{S^1} \phi \right) \left( \int_{S^1} \psi \right) \right| \leq c(\phi, \psi) \theta^n
\]

(hint: use Fourier analysis).

(3) Let \( T : X \to X \) be a continuous map of a topological space \( X \). Assume that \( X \) has a countable basis for open sets, and \( X \) is equipped with a probability measure \( \mu \) of full support (this means that \( \mu(U) > 0 \) for every open set \( U \subset X \)). Show that if \( T \) is mixing, then for a set of full measure in \( X \), the orbit \( \{T^n x\}_{n \geq 0} \) is dense.

(4) Prove that every number of the form \( a + b\sqrt{d} \) where \( a, b \in \mathbb{Z} \) and \( d \in \mathbb{N} \setminus \mathbb{N}^2 \) is badly approximable.

(5) Let \( \psi : [1, \infty) \to (0, 1) \) be any decreasing function. Show that there exist irrational numbers which are \( \psi \)-approximable.

(6) (quadratic irrationals ↔ periodic orbits) Let \( d \in \mathbb{N} \) and \( (x, y) \in \mathbb{Z}^2 \) be a solution of the Pell equation \( x^2 - dy^2 = 1 \). Show that every such solution gives rise to a periodic orbit of the flow

\[
a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}
\]

on \( \text{SL}_2(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{R}) \) with period \( \cosh^{-1}(x) \).

Namely, construct \( z \in \text{SL}_2(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{R}) \) such that \( za_{t_0} = z \) for \( t_0 = \cosh^{-1}(x) \).

(7) (a) Show that for every \( g \in \text{SL}_2(\mathbb{Z}) \) and \( h \in \text{SL}_2(\mathbb{Q}) \), there exists \( n \in \mathbb{N} \) such that \( h^{-1} g^n h \in \text{SL}_2(\mathbb{Z}) \).

(b) Show that if the orbit of \( a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \) for the point

\[
z_0 = \text{SL}_2(\mathbb{Z}) g_0 \in \text{SL}_2(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{R})
\]

is periodic, then so is the orbit for \( z = \text{SL}_2(\mathbb{Z}) h g_0 \) for every \( h \in \text{SL}_2(\mathbb{Q}) \).

(c) Deduce that the periodic orbits of the flow \( a_t \) in \( \text{SL}_2(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{R}) \) are dense.

(8) An element \( \gamma \in \text{SL}_2(\mathbb{Z}) \) is called \textit{primitive} if it cannot be written as \( \gamma = \gamma_0^m \) for some \( \gamma_0 \in \text{SL}_2(\mathbb{Z}) \) and \( m \in \mathbb{N}, m \geq 2 \). Show that every element of infinite order in \( \text{SL}_2(\mathbb{Z}) \) is a power of a primitive element.
(9) Prove that there is a one-to-one correspondence between periodic orbits of the flow \( a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \) on \( \text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R}) \) and conjugacy classes of primitive elements \( \gamma \in \text{SL}_2(\mathbb{Z}) \) with \( \text{Tr}(\gamma) > 2 \). Show that this correspondence the periods of the orbits are given by \( \cosh^{-1}(\text{Tr}(\gamma)/2) \).

(10) (singular vectors ↔ divergent trajectories) A vector \( x \in \mathbb{R}^d \) is called singular if for every \( \varepsilon > 0 \) and \( N \geq N_0(\varepsilon) \), the system of inequalities

\[
\left\| x - \frac{p}{q} \right\| < \frac{\varepsilon N^{-1/d}}{q}, \quad 0 < q < N
\]

has a solution \( p \in \mathbb{Z}^d \) and \( q \in \mathbb{N} \). As in the lectures, we use notation:

\[
\Lambda_x = \mathbb{Z}^{d+1} \begin{pmatrix} id & 0 \\ x & 1 \end{pmatrix} \in \mathcal{L}_{d+1}^d,
\]

\[
a_t = \text{diag}(e^t, \ldots, e^t, e^{-dt}) \in \text{SL}_{d+1}(\mathbb{R}).
\]

(a) Prove that a vector \( x \in \mathbb{R}^d \) is singular if and only if the orbit \( \{\Lambda_x a_t\}_{t \geq 0} \) is divergent (that is, \( \Delta(\Lambda_x a_t) \to 0 \) as \( t \to \infty \)).

(b) Deduce that the set of singular vectors in \( \mathbb{R}^d \) has Lebesgue measure zero.