Lecture 4

Diophantine approximation & flows.

For \( x \in \mathbb{R}^d \), we would like to find a rational approximation:
\[ x \approx \frac{p}{q} \in \mathbb{Q}^d. \]

**Thm (Dirichlet)** \( \forall x \in \mathbb{R}^d \ \forall R > 1 \ \exists p \in \mathbb{Z}^d, q \in \mathbb{N} : 
\|x - \frac{p}{q}\| \leq \frac{R^{-1/d}}{q} \]
\[ 1/q \leq R. \]

Can one improve Dirichlet’s Thm?

**Ex.** \( x = \sqrt{2} \) We know that \( |\sqrt{2} - \frac{p}{q}| \leq \frac{1}{q^2} \) has infinitely many solutions. Suppose that for some \( c \in (0,1) \),
\( |\sqrt{2} - \frac{p}{q}| \leq \frac{c}{q^2} \) also has infinitely many solutions.

Then \[ \frac{1}{q^2} \leq \frac{|2q^2-p^2|}{q^2} = |\sqrt{2} - \frac{p}{q}| \cdot |\sqrt{2} + \frac{p}{q}| \]
\[ \leq \frac{c}{q^2} \cdot \left( \frac{c}{q^2} + 2\sqrt{2} \right). \]

Hence, \( 1 \leq c \cdot (\frac{c}{q^2} + 2\sqrt{2}) \), and taking \( q \to \infty \), we conclude that \( c \geq \frac{1}{2\sqrt{2}}. \)

**Def** \( x \in \mathbb{R}^d \) is called **badly approximable** if
\[ \exists c > 0: \forall p \in \mathbb{Z}^d \ \forall q \in \mathbb{N} : \|x - \frac{p}{q}\| > \frac{c}{q^{1+d}}. \]

**Open question:** Is \( 3\sqrt{2} \) badly approximable?
Topology on the space of lattices:

We say that \( \Lambda_n \to \Lambda \) for lattices \( \Lambda_n, \Lambda \subseteq \mathbb{R}^d \) if for some (every) basis \( \{u_n^{(i)}\} \) of \( \Lambda_n \), \( \{u_n^{(i)}\} \) converges to a basis of \( \Lambda \).

Remark: The space \( \mathbb{L}_d' \) of unimodular lattices is not compact (e.g., \( \Lambda_n = \langle \frac{1}{n}e_1, n\cdot e_2, e_3, \ldots, e_d \rangle \) has no convergent subsequence).

Mahler compactness criterion:

\( \Omega \subseteq \mathbb{L}_d' \) is precompact \( \iff \exists S > 0 \): \( \forall \Omega \subseteq \mathbb{L} \): \( \forall x \in \Omega : \|x\| \leq S \).

(bounded)

Suppose that \( \Omega \) is precompact, but \( \exists \lambda_n \in \Omega : x_n \to 0 \).

After passing to a subsequence, we may assume that \( \Lambda_n \to \Lambda \). Let \( \Lambda_n = \langle u_n^{(i)} \rangle \) and \( \Lambda = \langle u^{(i)} \rangle \) with \( u_n^{(i)} \to u^{(i)} \).

Then \( x_n = \sum_i \lambda_n^{(i)} u_n^{(i)} = \sum_i \beta_u^{(i)} u^{(i)} \) and \( \beta_u^{(i)} \to 0 \).

This implies that \( \Omega_n \to 0 \), and since \( \lambda_n^{(i)} \in \mathbb{Z} \), \( \lambda_n^{(i)} = 0 \).

Recall that \( SL_d(\mathbb{R}) = SL_d(\mathbb{Z}) \cdot \Sigma_{tv} \), where \( \Sigma_{tv} = U_v A_v K_v \) is a Siegel set.

Hence, \( \mathbb{L}_d' = \mathbb{Z}^d \cdot \Sigma_{tv} \).

Let \( \Omega \subseteq \mathbb{L}_d' \) be such that \( \forall \Omega \subseteq \mathbb{L} : \forall x \in \Omega : \|x\| \leq S > 0 \).

Then \( \Omega = \mathbb{Z}^d \cdot \Sigma \) for some \( \Sigma \subseteq \Sigma_{tv} \).

For \( \sigma \in \mathbb{Z} \), \( a(\sigma) \leq t \) and \( \|e_i - \sigma\| = a(\sigma) \geq S \).
Hence, $\alpha(g)_i \geq t^{-1} \cdot \alpha(g)_{i-1} \geq \ldots \geq t^{-(i-1)} \cdot S$.

On the other hand, $\det(g) = \alpha(g)$, ..., $\alpha(g)_d = 1$, and each $\alpha(g)_i$ is uniformly bounded from above.

We have shown that $\exists c, C > 0: \forall g \in \Sigma: c \leq \alpha(g)_i \leq C$, so that $\Sigma$ is bounded in $SL_d(R)$, and hence precompact.

\textbf{Dani correspondence.}

For $x \in R^d$, set $u(x) = \left( \begin{array}{c} \text{Id} \\ 0 \\ x \\ 1 \end{array} \right)$ and $\Lambda_x = \mathbb{Z}^d u(x) \in L_{d+1}^1$.

Consider the flow $a_t: L_{d+1}^1 \to L_{d+1}^1$ where $a_t = \left( \begin{array}{cc} e^{t \text{Id}} & 0 \\ 0 & e^{-dt} \end{array} \right)$.

\textbf{Thm (Dani)} For every $x \in R^d$,

\[ x \text{ is badly approximable } \iff \text{the orbit } \{ \Lambda_x a_t \}_{t \geq 0} \text{ is bounded in } L_{d+1}^1. \]

Note that $\Lambda_x = \mathbb{Z}^d u(x) = \{ (p + g x, g) : p \in \mathbb{Z}^d, g \in \mathbb{Z} \}$.

By Mahler compactness criterion,

\[ \{ \Lambda_x a_t \}_{t \geq 0} \text{ is bounded in } L_{d+1}^1 \iff \exists S \in (0,1): \max \{ e^t \| p + g x \|, e^{-dt} \| g \| \} \geq S \]

for all $(p, g) \in \mathbb{Z}^d \times \{0\}$ and $t \geq 0$.

\[ \forall S \in (0,1): \max \{ e^t \| p - q x \|, e^{-dt} \| q \| \} \geq S \]

for all $p \in \mathbb{Z}^d$, $q \in \mathbb{N}$, $t \geq 0$. 
Suppose that $x$ is badly approximable, i.e.,
$$\|x - \frac{p}{q}\| \geq \frac{c}{q^{1+d}}$$
for all $p \in \mathbb{Z}^d$ and $q \in \mathbb{N}$.

Then
$$\frac{1}{g^{1+d}} \cdot \|p-gx\| = \left(\frac{e^{-dt}}{g^{1+d}}\right)^{1/d} \cdot \left(\frac{e^{t}\|p-gx\|}{e^{g^{1+d}}}\right) \geq c,$$
and
$$\max \left\{ e^{t}\|p-gx\|, e^{-dt} \right\} \geq c^{(1+d)^{-1}}.$$

Hence, $\{\lambda_n a_t\}_{t \geq 0}$ is bounded in $L_{d+1}$.

Conversely, suppose that $\{\lambda_n a_t\}_{t \geq 0}$ is bounded in $L_{d+1}$, i.e.,
for some $\varepsilon \in (0,1)$:
$$\max \left\{ e^{t}\|p-gx\|, e^{-dt} \right\} \geq \varepsilon,$$
for all $p \in \mathbb{Z}^d$, $g \in \mathbb{N}$, $t \geq 0$.

We pick $t > 0$ such that $e^{-dt} = \varepsilon/2$.

Then
$$\left(\frac{\varepsilon}{2} \cdot g^{1+d}\right)^{1/d} \cdot \|p-gx\| \geq \varepsilon$$
for all $p \in \mathbb{Z}^d$, $g \in \mathbb{N}$.

$$\|x - \frac{p}{g}\| \geq \frac{\varepsilon \cdot (\varepsilon/2)^{1/d}}{g^{1+d}} \Rightarrow x \text{ is badly approximable}.$$  

Cor. The set of badly approximable vectors in $\mathbb{R}^d$ has measure zero.

It follows from Howe–Moore Thm that for a set of full measure in $SL_d(\mathbb{R})$, the orbit $\{\mathbb{Z}^d g a_t\}_{t \geq 0}$ is dense in $L_{d+1}$ (exercise).

In particular, the set $\{g \in SL_{d+1}(\mathbb{R}) : \{\mathbb{Z}^d g a_t\}_{t \geq 0} \text{ is bounded}\}$ has measure zero.

Now
$$\{g \in SL_d(\mathbb{R}) : M_{dd} \neq 0\} = \left(\frac{\text{Id} | 0\}_{d \times d}}{\mathbb{R}^d} \right) \cdot \left(\begin{array}{c|c}
\ast & \ast \\
\hline
\ast & \ast
\end{array}\right)$$

(d-1)-minor
Invariant measure on $S^d_{+1}(\mathbb{R}^d)$:

$$
\int_{S^d_{+1}(\mathbb{R}^d)} \mathbf{f}(x) \, d\mu(g) = \int_{\mathbb{R}^d \times \mathcal{B}} \mathbf{f}(u(x)b) \, dx \, dp(b)
$$

Lebesgue measure on $\mathbb{R}^d$.

We have $\mathbb{Z}^{d+1} u(x), b a_t = (\mathbb{Z}^{d+1} u(x) a_t) \cdot (b a_t) a_t$.

$$
a_t^{-1} \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} a_t = \begin{pmatrix} B_{11} & \tilde{c}(d+t) B_{12} \\ 0 & B_{22} \end{pmatrix} - \text{bounded for } t \geq 0
$$

Hence, $\{ \mathbb{Z}^{d+1} u(x) b a_t \}_{t \geq 0}$ is bounded $\iff \{ \mathbb{Z}^{d+1} u(x) a_t \}_{t \geq 0}$ is bounded,

and $\{ g \in u(\mathbb{R}^d) : \{ \mathbb{Z}^{d+1} u(x) a_t \}_{t \geq 0} \text{ is bounded} \}$

$\leftarrow$ Dani's Thm.

$\{ u(x) : \{ \mathbb{Z}^{d+1} u(x) a_t \}_{t \geq 0} \text{ is bounded} \} \cdot \mathcal{B}$.

$\{ u(x) : x \in \mathbb{R}^d - \text{badly approximable} \} \cdot \mathcal{B}$.

Since this set has measure zero, this implies the claim.