Schmidt game.
Fix a subset $S$ of $\mathbb{R}^d$ and parameters $\alpha, \beta \in (0,1)$.

2 players: Alice and Bob.

- Bob picks a (closed) cube $B_1 \subset \mathbb{R}^d$.
- Alice picks a cube $A_1 \subset B_1$ of size $\alpha \cdot \text{size}(B_1)$.
- Bob picks a cube $B_2 \subset A_1$ of size $\alpha \beta \cdot \text{size}(B_1)$.
- Alice picks a cube $A_2 \subset B_2$ of size $\alpha^2 \beta \cdot \text{size}(B_1)$.

At the end, $\bigcap_{n=1}^\infty A_n \cap \bigcap_{n=1}^\infty B_n = \{x\}$.
Alice wins if $x \in S$.

Def. 1) $S \subset \mathbb{R}^d$ is called $(d, \beta)$-winning if Alice can design a strategy so that she can always win, regardless of moves Bob chooses.

2) $S$ is called $\alpha$-winning if it is $(d, \beta)$-winning for every $\beta \in (0, \beta_0)$.

Winning sets are BIG.

Def. $f: \mathbb{R}^d \to \mathbb{R}^d$ is called $c$-bi-Lipschitz if
$$c' \|x_1 - x_2\| \leq \|f(x_1) - f(x_2)\| \leq c \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in \mathbb{R}^d.$$
Lemma 1. If $S \subseteq \mathbb{R}^d$ is $\alpha$-winning, then $f(S)$ is $(c^2 \alpha)$-winning.

Proof.

At every step of the game $f(S)$, Alice can exploit the winning strategy of the game $S$ as follows. As Bob picks a cube $B_i$ of size $b$, $f'(B_i)$ contains a cube of size $c^2 b$, and Alice can pick a cube $A' \subseteq f'(B_i)$ of size $c^2 b \alpha$ which realizes the winning strategy for $S$. Then $f(A'_i)$ contains a cube $A_i$ of size $c^2 b \alpha$.

If we continue this way, $\bigcap_{n=1}^{\infty} A'_i \subseteq S$. Hence $\bigcap_{n=1}^{\infty} A'_i \subseteq f(S)$.

Lemma 2. If $S_i, i \geq 1$, are $\alpha$-winning sets, then so is $\bigcap_{i=1}^{\infty} S_i$.

Proof. Alice can play winning strategies for the games $S_i$ simultaneously: at step $n = 2^{i-1} + 2^i (k-1)$, $k \geq 1$, Alice applies the winning strategy for the game $(\alpha, \beta (\alpha \beta)^{2^i})$ with the end set $S_i$. 


Lem 3. Every $\alpha$-winning set has cardinality continuum.

(In fact, every $\alpha$-winning set has full Hausdorff dimension.)

Proof. If $B < \frac{1}{\alpha}$, Bob can pick at every step of the game a cube contained in the upper half or $A_n$ or in the lower half of $A_n$.

Hence, Bob can pick any path on the above tree at his will. Since different paths lead to different points in $\mathbb{R}^d$, the cardinality of a winning set should be at least the cardinality of paths in the above tree.

Thm. The set $BA$ of badly approximable vectors in $\mathbb{R}^d$ is winning.

Cor. For every $c$-bilipschitz maps $f_i$,

$\bigcap_{i \geq 1} f_i(BA)$ has full Hausdorff dimension.
Proof of Thm. (for $d = 1$).

We start with an elementary lemma:

**Lem.** $\exists \alpha, \eta \in (0,1); Vw \in \mathbb{R}^2: \exists$ interval $A' \subset [-\frac{1}{2}, \frac{1}{2}]: |A'| = \alpha \eta$

such that $Vw \in A': 1w_1 + xw_2 \geq \eta \cdot \|w\|_1$.

Recall that by Domi Thm (Lecture 1):

$x \in BA \iff \{2^2 u(x)q_n\}_{n \geq 0}$ is bounded in $\ell_2$. (+)

Fix $\alpha \in (0, \alpha_0)$ and $\beta \in (0, 1)$.

Given an interval $B_{i_1}, |B_{i_1}| = \beta$, which is initial pick
of Bob, we need to provide a winning strategy
for Alice. Note that

$(+) \iff \{2^2 u(x)q_n\}_{n \geq 1}$ is bounded

where $q_n = \begin{pmatrix} b^{-\frac{1}{2}} (\alpha \beta)^{\frac{3}{2}} & 0 \\ 0 & b^\frac{1}{2} (\alpha \beta)^{\frac{1}{2}} \end{pmatrix}$.

We shall give a bounded set $S \subset \ell_2$ and
a strategy $\{A_n\}_{n \geq 1}$ for Alice such that

$\forall x \in A_n: 2^2 u(x)q_n \in S$.

Then $\{x_{\infty}\} = \bigcap_{n \geq 1} A_n$ satisfies

$2^2 u(x_{\infty})q_n \in S$ for all $n$, so
that $x_{\infty} \in BA$, and Alice wins.

Recall:

**Exercise 1 (Lecture 1)**

$\exists \epsilon > 0: \forall A \subset \ell_2: \bigwedge_{\theta \in (0, \epsilon)} \exists$ at most one (up to sign) primitive vector.
We pick $S \in \mathcal{C}(x_i)$ such that

\((*) \quad \forall x \in B_1 : \quad \exists z \in (x \cap B_1) \cap B_1 : \quad \exists \gamma \in \mathbb{R} : \quad \|x - \gamma \cdot z\| < \varepsilon_0,\)

\((** \quad \forall x \in [-a_x, a_x]^n] : \forall s \in \mathbb{R}^2 : \|s \cdot u(x)\| < \varepsilon_0 \Rightarrow \|s\| < \varepsilon_0. \)

Let $S_2 = \{ \lambda \in S_2 : \forall x \in \Lambda : \|x\| > 2S_1 \}.$

**Main Claim:** Alice can always pick an interval $A_n \subset B_n : \|A_n\| = \alpha \|B_n\|$ such that $\forall x \in A_n, \forall \text{primitive } v \in \mathbb{Z}^2,$ either:

1. $\|v \cdot u(x)g_n\| \geq \delta,$
2. $\|v \cdot u(x)g_n\| > 2\delta.$

Note that this claim will finish the proof.

We use induction on $n.$

For $n=1,$ it follows from $(*)$ that (1) holds for every $x \in B_1,$ so Alice can pick any $A_1 \subset B_1.$

**Subclaim:** (1) may fail for at most one (up to sign) primitive $v \in \mathbb{Z}^2.$

Suppose that we have $v_1, v_2 \in \mathbb{Z}^2, v_1 \neq v_2, x_1, x_2 \in B_n$ such that $\|v_i \cdot u(x)g_n\| < \delta,$ $i=1, 2.$

Then $v_2 \cdot u(x_1)g_n = v_2 \cdot u(x_2)g_n \cdot (g_n^{-1} \cdot u(x_1 - x_2))g_n$

$= v_2 \cdot u(x_2)g_n \cdot u(b^{-1}(\alpha \beta)^{-1}(x_1 - x_2))$

Since $\|B_n\| = b(\alpha \beta)^{-1}, \quad b^{-1}(\alpha \beta)^{-1}(x_1 - x_2) \in [-a_x, a_x].$

Hence, by $(**), \quad \|v_2 \cdot u(x)g_n\| < \delta.$

Since also $\|v_i \cdot u(x)g_n\| < \delta < \varepsilon_0,$ we get a contradiction.
Now suppose that the claim holds for \( n-1 \). Then for every \( x \in B_n \), and primitive \( v \in \mathbb{Z}^2 \), either:

1. \( \| v u(x) g_{n-1} \| \geq \varepsilon \),

2. \( \| (v u(x) g_{n-1}) \| \geq \varepsilon \).

(2) \Rightarrow \| (v u(x) g_n) \| = (d \beta)^{-1} \cdot \| (v u(x) g_{n-1}) \| \geq \varepsilon \).

If \( \| v u(x) g_n \| \geq \varepsilon \) fails for some primitive \( v \in \mathbb{Z}^2 \), it can fail for at most one vector (up to \( \text{sign} \)) — \( \varepsilon \).

We have \( B_n = b (d \beta)^{-n} [-\frac{1}{2}, \frac{1}{2}] + x_n \) for \( x_n \in B_n \).

\[ x = b (d \beta)^{-n} y + x_n \text{ for } y \in [-\frac{1}{2}, \frac{1}{2}] . \]

\[ v u(x) g_{n-1} = v u(x_n) g_{n-1} \cdot g_{n-1}^{-1} u( b (d \beta)^{-n} y ) g_{n-1} \]

\[ = v u(x_n) g_{n-1} \cdot u(y) . \]

Pick \( A' \in [-\frac{1}{2}, \frac{1}{2}] \), \( |A'| = \alpha \), according to Lem., and set \( A_n = b (d \beta)^{-n} A' + x_n \).

Then for every \( x = b (d \beta)^{-n} y + x_n \in A_n \), we have

\[ \| (v u(x) g_n) \| \geq \| (v u(x) g_{n-1}) \| = \| (v u(y)) \| \]

\[ \geq |w_1 + y w_2| \geq \varepsilon \| w \| = \varepsilon \| u(x_n) g_{n-1} \| \geq \varepsilon \| y \| \text{ (Lem.)} \]

Hence, (2) holds for "bad" \( \varepsilon \).

This proves the claim.