Let $G$ be a closed subgroup of $SL_2(\mathbb{R})$, and $\Gamma$ is a discrete subgroup such that $\Gamma \backslash G$ has finite measure.

A one-parameter subgroup $U = \{ u(t) \} \subset G$ is called unipotent if $u(t) = \exp(t \cdot N)$ where $N$ is a nilpotent matrix.

(Here, $\exp(x) = I + x + \frac{x^2}{2!} + \ldots$).

Our interest is the dynamical system:

$$X = \Gamma \backslash G \overset{\{ u(t) \}}{\longrightarrow} x \longmapsto x \cdot u(t),$$

which is a generalisation of the horocycle flow.

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**Thm (Ratner)**

1) **(topological version)** For every $x \in X$, $xU = xF$,

where $F$ is a closed connected subgroup of $G$, containing $U$, and $xF$ supports finite $F$-invariant measure.

More generally, if $H$ is a closed subgroup of $G$ generated by unipotent subgroups, then

$$\forall x \in X: x \cdot H = xF$$

where $F$ is a closed connected subgroup of $G$, containing $H$, and $xF$ supports finite $F$-inv. measure.
2) (measurable version) Every ergodic $U$-invariant probability measure on $X$ is an $F$-invariant probability measure supported on closed orbit $xF$, $x \in X$.

3) (equidistribution) For every $x \in X$, 
\[
\frac{1}{T} \int_0^T \delta_{F(xu_t)} dt \xrightarrow{T \to \infty} \int_X \delta_{x} d\mu,
\]
where $\mu$ is the $F$-invariant probability measure supported on $xU = xF$.

**Oppenheim Conjecture.**

\[ Q(x_1, \ldots, x_d) = \frac{d}{\sum_{i,j=1}^{d} a_{ij} x_i x_j}, \quad a_{ij} \in \mathbb{R}, \quad a_{ij} = a_{ji}. \]

Assume that: 1) $Q$ is nondegenerate ($\iff \det(a_{ij}) \neq 0$), 2) $Q$ is indefinite ($\iff Q(\mathbb{R}^d) = \mathbb{R}$), 3) $Q$ is not a scalar multiple of a form with rational coefficients.

**Ex.** $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - \sqrt{2} x_3^2$.

**Conj. (Oppenheim)** If $d \geq 3$, then $Q(\mathbb{Z}^d)$ is dense in $\mathbb{R}$.

This conjecture was proved by Margulis.

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**Ratner Thm $\Rightarrow$ Oppenheim Conj. (sketch)**

Let $SO(Q, \mathbb{R}) = \{ g \in SL_d(\mathbb{R}) : Q(x, g) = Q(x) \text{ for all } x \}$.

Let $H = \text{the connected component of } SO(Q, \mathbb{R}) \text{ containing identity.}$
Algebraic properties: when $d \geq 3$,

1) $H$ is generated by unipotent subgroups.
2) $H$ is maximal connected subgroup of $\text{SL}_d(\mathbb{R})$.

By the Ratner Thm, we have two cases:

- $\mathbb{Z}^d H = \{ \text{whole space} \}$
- $\mathbb{Z}^d H$ is closed

\[ \text{SL}_d(\mathbb{Z}) H \text{ is dense in } \text{SL}_d(\mathbb{R}) \]

$Q(\mathbb{Z}^d) = Q(\mathbb{Z}^d, \text{SL}_d(\mathbb{Z}) H)$

\[ Q\left(\frac{\mathbb{Z}^d}{\text{SL}_d(\mathbb{R})}\right) \]

\[ Q(\mathbb{R}) \]

Let $\Gamma = H \cap \text{SL}_d(\mathbb{Z})$.

Then $\text{vol}(\Gamma \backslash H) < \infty$.

(\(\Gamma\) is "large")

Consider the system of linear equations:

\[ \{ A^* A = A, \quad h A h = A : h \in H \} \]

One can check that the set of solutions is $\langle A_q \rangle$

where $A_q$ is the matrix of the quadratic form $Q$.

We also consider the system of linear equations:

\[ \{ A^* A = A, \quad h A h = A : h \in \Gamma \} \]

which has integral coefficients. Hence, its set of solutions is a rational subspace of $\text{Mat}_d(\mathbb{R})$.

Using that $\Gamma$ is "large", one can show that

(\(\ast\)) and (\(\ast\ast\)) have the same sets of solutions.

This implies that $A_q$ is a multiple of a rational matrix.
Sprindzuk Conjecture.

A vector \( x \in \mathbb{R}^d \) is called **well approximable** if
\[
\| x - \frac{q}{p} \| < \frac{1}{q^{1+\delta}} + \epsilon, \quad p \in \mathbb{Z}^d, \quad q \in \mathbb{N},
\]
has infinitely many solutions for some \( \epsilon > 0 \).

By the Borel-Cantelli lemma, the set \( \text{WA}_d \) of well approximable vectors in \( \mathbb{R}^d \) has measure 0.

Let \( U \subseteq \mathbb{R}^k \) be open and \( f: U \to \mathbb{R}^d \) be polynomial map.

**Conj. (Sprindzuk)** Assume that \( f(U) \) is not contained in a proper affine subspace of \( \mathbb{R}^d \). Then \( \{ x \in U : f(x) \in \text{WA}_d \} \) has measure zero.

This conjecture was proved by Kleinbock & Margulis using "nondivergence" properties of unipotent flows.

Let \( \Lambda_x = \mathbb{Z}^d \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right\} \), \( \alpha = \begin{pmatrix} e^{t \text{Id}} & 0 \\ 0 & e^{-dt} \end{pmatrix} \), \( \Delta(\lambda) = \min\{ \| \lambda \| : \lambda \in \text{Vol} \} \).

**Lemma.** \( x \in \text{VA}_d \iff \exists \delta > 0 : \forall n \to \infty: \Delta(\Lambda_x^{an}) \leq e^{-\delta n} \).

(excursions to exponentially shrinking nbhds of infinity).
Prop. \( \forall x \in U \exists a \text{ ball } x \in B \subset U \text{ such that } \forall \varepsilon, t > 0: \)
\[
\mu \left( \{ x \in B : \Delta (N, a^t) < \varepsilon \} \right) \leq C \cdot \varepsilon^x \cdot \text{vol}(B).
\]

\( \mu(B, a^t) \) \( \uparrow \) The percentage of time spend near infinity can be controlled.

Prop. \( \Rightarrow \) Conj. \( \{ x \in B : f(x) \in V A_d \} = \bigcup_{S > 0} \{ x \in B : \Delta (N, f(a^S n)) \leq \varepsilon^S \} \) for inf. many \( n \in N \).

It is sufficient to show that \( \text{vol}(\mathcal{J}_S) = 0 \) for all \( S > 0. \)

Let \( \mathcal{J}_S(n) = \{ x \in B : \Delta (N, f(a^n)) \leq \varepsilon^S \} \).

Then \( \mathcal{J}_S = \lim_{n \to \infty} \mathcal{J}_S(n) = \bigcap_{n \geq 1} U \mathcal{J}_S(n) \), and

\[
\text{vol}(\mathcal{J}_S) \leq \sum_{n \geq 1} \text{vol}(\mathcal{J}_S(n)) \leq \sum_{n \geq 1} \frac{C \cdot (\varepsilon^S)^n}{\text{vol}(B)} \xrightarrow{N \to \infty} 0.
\]

\( \text{Def.} \) A function \( f : U \to \mathbb{R} \) is called \( (C, \varepsilon, \alpha) \)-good if for any ball \( B \subset U \) and any \( \varepsilon > 0, \)

\[
\text{vol}(\{ x \in B : |f(x)| < \varepsilon \cdot \sup_{x \in B} |f(x)| \}) \leq C \cdot \varepsilon^\alpha \cdot \text{vol}(B).
\]

\( \text{Lem.} \) \( \forall d \geq 1 \exists (C, \varepsilon, \alpha) : \) A polynomial \( f : \mathbb{R} \to \mathbb{R} \) of degree \( d \) is \( (C, \varepsilon, \alpha) \)-good.
Without loss of generality, \( \sup_{x \in B} |f(x)| = 1 \).

Let \( B_\varepsilon = \{ x \in B : |f(x)| < \varepsilon \} \) and \( \ell = |B_\varepsilon| \).

Since \( B_\varepsilon \) cannot be covered by \((d-1)\) intervals of length \( \ell/d \), one can find \( x_1, \ldots, x_{d+1} \in B_\varepsilon \) for \( i \neq j \).

By the Lagrange Interpolation,

\[
    f(x) = \sum_{i=1}^{d+1} f(x_i) \cdot \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}. 
\]

\( \forall x \in B : |f(x)| \leq (d+1) \cdot \varepsilon \cdot \frac{|B|}{(2^d)^d} \).

Hence, \( 1 \leq (d+1) \cdot \frac{|B|}{(2^d)^d} \Rightarrow \ell \leq C_d \cdot \varepsilon^\frac{1}{d} \cdot |B| \).

To prove Proposition, we need to estimate:

\[
    \text{vol}\left( \{ x \in B : \Delta Rf(x) a_+ < \varepsilon \} \right).
\]

For \( v \in \Lambda \), set \( P_{v_+}(x) = \| v \cdot \left( \frac{\tilde{x}}{\|\tilde{x}\|_1} \right) a_+ \|_1 \), \( x \in B \).

By the \((C, \alpha)\)-good property,

\[
    \text{vol}\left( \{ x \in B : P_{v_+}(x) < \varepsilon \} \right) \leq C \cdot \left( \frac{\varepsilon}{\sup_B (P_{v_+})} \right)^\alpha \cdot |B|.
\]

This is the first step towards the proof.

However, \( \bigcup_{0 + v \in \Lambda} \{ x \in B : \Delta Rf(x) a_+ < \varepsilon \} \)

so that one needs additional combinatorial arguments to deduce the estimate.