9 Modes of convergence

9.1 Weak convergence in normed spaces

We recall that the notion of convergence on a normed space $X$, which we used so far, is the convergence with respect to the norm on $X$: namely, for a sequence $(x_n)_{n \geq 1}$, we say that $x_n \to x$ if

$$\|x_n - x\| \to 0 \quad \text{as } n \to \infty.$$  

However, in many cases, this notion does not capture the full information about behaviour of sequences. Moreover, we recall that according to the Bolzano–Weierstrass Theorem, a bounded sequence in $\mathbb{R}$ has a convergent subsequence. An analogue of this result fails for in infinite dimensional normed spaces if we only deal with convergence with respect to norm.

Example 9.1. Consider the sequence $e_n = (0, \ldots, 0, 1, 0, \ldots)$ in the space $\ell^2$. Then for $n \neq m$, $\|e_n - e_m\|_2 = \sqrt{2}$. So that we conclude that the sequence $(e_n)_{n \geq 1}$ is not Cauchy, so it cannot converge, and it even does not contain any convergent subsequence. Nonetheless, in some “weak” sense, we may say that this sequence “converges” to $(0, \ldots, 0, \ldots)$. For instance, for every $x = (x_n)_{n \geq 1} \in \ell^2$,

$$\langle x, e_n \rangle = x_n \to 0.$$

This example motivates the notion of weak convergence which we now introduce.

Definition 9.2. A sequence $(x_n)_{n \geq 1}$ in a normed space $X$ converges weakly to $x \in X$ if for every $f \in X^*$, we have that $f(x_n) \to f(x)$. We write

$$x_n \overset{w}{\to} x.$$

To emphasise the difference with the usual notion of convergence, if (1) hold, we say $(x_n)_{n \geq 1}$ converges in norm or converges strongly.

Returning to Example 9.1, we see that the sequence $(e_n)_{n \geq 1}$ converges weakly, but has no subsequences that converge strongly. Indeed, any $f \in (\ell^2)^*$ is of the form $f(y) = \langle y, x \rangle$ for some $x \in \ell^2$, and $\langle e_n, x \rangle = x_n \to 0$.

Here is another example of a sequence which converges weakly, but not strongly.

Example 9.3. Let $X$ be the space of real-valued continuous functions with the max-norm, and

$$\phi_n(t) = \begin{cases} nt & \text{when } 0 \leq t \leq 1/n, \\ 2 - nt & \text{when } 1/n \leq t \leq 2/n, \\ 0 & \text{when } 2/n \leq t \leq 1. \end{cases}$$

1
We claim that $\phi_n \overset{w}{\to} 0$. Suppose that this is not the case. Then there exists $f \in X^*$ such that $f(\phi_n) \not\to 0$. Passing to a subsequence we may assume that $|f(\phi_n)| \geq \delta$ for some fixed $\delta > 0$, and without loss of generality

$$f(\phi_n) \geq \delta.$$ 

Moreover, we may assume that the subsequence satisfies $n_{i+1} \geq 2n_i$. Let

$$\psi_N = \sum_{i=1}^{N} \phi_{n_i}$$

We have

$$\psi_N(t) \leq \sum_{i: n_i \leq 1/t} n_it + \sum_{i: 1/t < n_i \leq 2/t} (2 - n_it).$$

Let $k = \max\{i : n_i \leq 1/t\}$. Then for all $i \leq k$,

$$n_i \leq n_k/2^{k-i} \leq 1/t \cdot 1/2^{k-i},$$

and the first sum satisfies

$$\leq \sum_{i \leq k} 1/2^{k-i} \leq 2.$$

To estimate the second sum we observe that because $n_{i+1} \geq 2n_i$, the inequality $1/t < n_i \leq 2/t$ may holds for at most one index $i$. Hence, the second sum is also bounded by 2. Hence,

$$\|\psi_N\|_{\infty} \leq 4 \quad \text{and} \quad |f(\psi_N)| \leq 4\|f\|.$$ 

On the other hand,

$$f(\psi_N) = \sum_{i=1}^{N} f(\phi_{n_i}) \geq N\delta \to \infty \quad \text{as} \quad N \to \infty.$$ 

This gives a contradiction.

Since $\|\phi_n\|_{\infty} = 1$, $\phi_n \not\overset{w}{\to} 0$ strongly.

We derive some basic properties of weak convergence.

**Theorem 9.4.** 1. If $x_n \to x$, then $x_n \overset{w}{\to} x$.

2. Weak limits are unique.

3. If $(x_n)_{n \geq 1}$ is a weakly convergent sequence, then the sequence of norms $\|x_n\|$ is bounded.
Proof. 1. This follows from the estimate $|f(x_n) - f(x)| \leq \|f\| \|x_n - x\|$ for every $f \in X^*$.

2. Suppose $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$. Then for every $f \in X^*$, we have $f(x_n) \to f(x)$ and $f(x_n) \to f(y)$. Hence, it follows that $f(x) = f(y)$ for all $f \in X^*$. So that by a corollary of the Hahn–Banach Theorem, $x = y$.

3. We would like to apply the Uniform Boundedness Principle. For this we have to interpret $x_n$'s as maps on a suitable space. We consider the mapping $X \to (X^*)^*$ defined by $x \mapsto g_x$, where $g_x(f) = f(x)$ for any $f \in X^*$. Since $|g_x(f)| = |f(x)| \leq \|x\| \|f\|$, this indeed defines a bounded linear functional on $X^*$ with $\|g_x\| \leq \|x\|$. Moreover, by a corollary of Hahn–Banach Theorem, there exists $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\| = 1$. This implies that

$$\|g_x\| = \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\} \leq \|x\|.$$ 

Hence, $\|g_x\| = \|x\|$.

For short, put $g_{x_n} := g_n$. Since for every $f \in X^*$, the sequence $f(x_n)$ converges, it is bounded, that is, $|f(x_n)| \leq c_f$ where $c_f$ is some positive constant independent of $n$. Thus, $|g_n(f)| \leq c_f$ for all $n \in \mathbb{N}$. We recall that the dual space $X^*$ is always a Banach space. The Uniform Boundedness Principle tell us that the sequence $g_n$ is uniformly bounded, meaning $\|g_n\| \leq c$, where $c > 0$ is independent of $n$. This implies the claim.

The following theorem gives a convenient criterion for weak convergence.

**Theorem 9.5.** A sequence $(x_n)_{n \geq 1}$ in a normed space $X$ converges weakly to $x$ provided that

(i) $\|x_n\|$ is uniformly bounded,

(ii) For every element $f$ in a dense subset $M \subset X^*$, we have $f(x_n) \to f(x)$.

Proof. According to (i), $\|x_n\| < c$ and $\|x\| < c$ for some fixed $c > 0$.

We would like to show that for any $f \in X^*$, we have $f(x_n) \to f(x)$. Let $\epsilon > 0$ and $\{f_j\} \subset M \subset X^*$ be a sequence with $f_j \to f$ in $X^*$. We obtain

$$|f(x_n) - f(x)| \leq |f(x_n) - f_j(x_n)| + |f_j(x_n) - f_j(x)| + |f_j(x) - f(x)|.$$ (2)
Since \( f_j \to f \), we can choose \( j \) large enough that
\[
\|f_j - f\| < \frac{\epsilon}{3c}.
\]
Also, since \( f_j(x_n) \to f_j(x) \), there is a number \( N \) such that
\[
|f_j(x_n) - f_j(x)| < \frac{\epsilon}{3}
\]
whenever \( n > N \). We can now bound (2)
\[
|f(x_n) - f(x)| \leq |f(x_n) - f_j(x_n)| + |f_j(x_n) - f_j(x)| + |f_j(x) - f(x)|
\leq \|f - f_j\|\|x_n\| + |f_j(x_n) - f_j(x)| + \|f_j - f\|\|x\|
\leq \frac{\epsilon}{3c}c + \frac{\epsilon}{3} + \frac{\epsilon}{3c}c < \epsilon
\]
whenever \( n > N \). This proves that \( x_n \xrightarrow{w} x \). \( \square \)

Remark 9.6 (weak convergence in Hilbert spaces). If \( X \) is a Hilbert space then the Riesz–Frechét Theorem tells us that a sequence \( (x_n)_{n \geq 1} \subset X \) is weakly convergent to \( x \in X \) if and only if \( \langle x_n, z \rangle \to \langle x, z \rangle \) for all \( z \in X \). Theorem 9.5 then tell us that we only need to check that \( \langle x_n, v \rangle \to \langle x, v \rangle \) for elements \( v \) of some basis of \( X \).

### 9.2 Convergence of sequences of functionals

Now we consider convergence of a sequence of linear functional \( f_n \in X^* \).

Definition 9.7 (weak* convergence). We say that a sequence \( (f_n)_{n \geq 1} \) weak* converges to \( f \in X^* \) if for every \( x \in X \) we have that \( f_n(x) \to f(x) \). This is denoted by \( f_n \xrightarrow{w^*} f \).

We note that since the dual space \( X^* \) is also a normed space, it also makes sense to talk about strong and weak convergence in \( X^* \). Namely:

- a sequence \( f_n \in X^* \) converges strongly to \( f \) if \( \|f_n - f\| \to 0 \).
- a sequence \( f_n \in X^* \) converges weakly to \( f \) if for every \( g \in (X^*)^* \), we have \( g(f_n) \to g(f) \).

In general, we have:

- strong convergence \( \Rightarrow \) weak convergence \( \Rightarrow \) weak* convergence
To see that the second arrow is true, we note that every \(x \in X\) defines an element \(g_x \in (X^*)^*\) such that
\[
g_x(f) = f(x).
\]
This defines a map \(X \to (X^*)^*\). We note that in general this map is not surjective and weak\(^*\) convergence does not imply weak convergence.

We illustrate the notion of weak\(^*\) convergence by some examples.

**Example 9.8.** Let \(X = C[-1, 1]\) be the space of continuous functions, and
\[
\rho_n(t) = \begin{cases} 
  n - n^2|t| & \text{when } -1/n \leq t \leq 1/n, \\
  0 & \text{otherwise}.
\end{cases}
\]
We consider the sequence functionals \(f_n : X \to \mathbb{C}\) defined by
\[
f_n(\phi) = \int_{-1}^{1} \phi(t)\rho_n(t)dt, \quad \phi \in C[-1, 1].
\]
We claim that \(f_n\) weak\(^*\) converges to \(f_0\) defined by \(f_0(\phi) = \phi(0)\). Indeed, using that \(\int_{-1}^{1} \rho_n(t)dt = 1\), we obtain
\[
|f_n(\phi) - f_0(\phi)| = \left| \int_{-1}^{1} \phi(t)\rho_n(t)dt - \int_{-1}^{1} \phi(0)\rho_n(t)dt \right| \\
\leq \int_{-1}^{1} |\phi(t) - \phi(0)|\rho_n(t)dt \\
= \int_{-1/n}^{1/n} |\phi(t) - \phi(0)|\rho_n(t)dt \\
\leq \max_{-1/n \leq t \leq 1/n} |\phi(t) - \phi(0)|.
\]
Hence, it follows from continuity of \(\phi\) that \(f_n(\phi) \to f_0(\phi)\). This proves that \(f_n \stackrel{w^*}{\to} f_0\).

Although we will not prove it here, it is the case that \(f_n \not\to f_0\).

**Example 9.9.** Let \(X = c_0 = \{x = (x_n)_{n \geq 1} \in \ell^\infty : x_n \to 0\}\).

We have met this space in homeworks. We recall that \(c_0^* \simeq \ell^1\). More explicitly, all elements of \(c_0^*\) are of the form
\[
y_g(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x \in c_0
\]
for some \( y \in \ell^1 \). We consider the sequence of linear functionals \( f_n = f_{e_n} \). Then for every \( x \in c_0 \),

\[
  f_n(x) = x_n \to 0.
\]

Hence, \( f_n \overset{w^*}{\to} f \).

We also recall that \((\ell^1)^* \simeq \ell^\infty\), and all elements of \((\ell^1)^*\) are of the form

\[
g_z(y) = \sum_{n=1}^{\infty} y_n z_n, \quad y \in \ell^1
\]

for some \( z \in \ell^\infty \). Since \( g_z(e_n) = z_n \not\to 0 \) in general. The sequence \( f_n \) does not converge weakly to 0.

**Example 9.10.** If \( H \) is a Hilbert space, the Riesz–Frechét Theorem tells us that \( H^* \simeq H \), and the map \( H \to (H^*)^* \) defined in (3) is an isomorphism. This implies that in Hilbert spaces weak and weak* convergences are the same.

The following result is a direct corollary of the Uniform Boundedness Principle.

**Theorem 9.11.** If the sequence \((f_n)_{n \geq 1}\) in \( X^* \) is weak* convergent, then the sequence \( \|f_n\| \) is bounded.

The following theorem is analogous to Theorem 9.5. It tell us a way to determine whether a given sequence in \( X^* \) is weak* convergent, without having to check the defining condition on all the elements of \( X \). Its proof also runs in parallel with the proof of Theorem 9.5.

**Theorem 9.12.** A sequence \((f_n)_{n \geq 1}\) in \( X^* \) is weak* convergent provided that

(i) The sequence \( \|f_n\| \) is bounded.

(ii) The sequence \( f_n(x) \) is Cauchy for every \( x \) in a dense subset \( M \subset X \).

**Proof.** Fix \( c > 0 \) such that \( \|f_n\| < c \) for all \( n \). Now, let \( x \in X \). We can find a sequence \( x_j \) in \( M \) such that \( x_j \to x \). Let \( \epsilon > 0 \). We will show that we can make \( |f_m(x) - f_n(x)| < \epsilon \) by taking large enough \( m, n \). We carry out the first bound:

\[
  |f_m(x) - f_n(x)| \leq |f_m(x) - f_m(x_j)| + |f_m(x_j) - f_n(x_j)| + |f_n(x_j) - f_n(x)|
\]

\[
\leq \|f_m\|\|x - x_j\| + |f_m(x_j) - f_n(x_j)| + \|f_n\|\|x_j - x\|.
\]
Since $x_j \to x$, we can fix $j$ large enough that $\|x - x_j\| < \frac{\epsilon}{3c}$. Since the sequence $f_n(x_j)$ is Cauchy, there is a number $N$ such that whenever $m, n > N$, we have $|f_m(x_j) - f_n(x_j)| < \frac{\epsilon}{3}$. The above expression is now bounded by

$$< \frac{\epsilon}{3c} + \frac{\epsilon}{3} + \frac{\epsilon}{3c} = \epsilon,$$

which proves that the sequence $f_n(x)$ is indeed Cauchy, and so converges.

We define $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in X$. Since $f_n$’s are linear, it follows that $f$ is also linear. For every $x \in X$, $|f_n(x)| \leq \|f_n\| \|x\| \leq c\|x\|$. Passing to the limit, we conclude that $|f(x)| \leq \|f\| \|x\| \leq c\|x\|$, so that $\|f\| \leq c$, and $f \in X^*$. We have $f_n \overset{w^*}{\to} f$.

The following theorem is a generalisation of the Bolzano–Weierstrass theorem.

**Theorem 9.13** (weak* compactness). Suppose that a normed space $X$ contains a countable dense subset. Then every bounded sequence $f_n \in X^*$ contains a weak* convergent subsequence.

**Proof.** Let $M = \{x_i\}_{i \geq 1}$ be a countable dense subset of $X$. Since $(f_n)_{n \geq 1}$ is a bounded sequence, $(f_n(x))_{n \geq 1}$ is also bounded for every $x \in X$. In particular, $(f_n(x_1))_{n \geq 1}$ is bounded, and by the Bolzano–Weierstrass Theorem, there is a subsequence $n_1(k)$ such that $(f_{n_1(k)}(x_1))_{k \geq 1}$ converges. Next, the sequence $(f_{n_1(k)}(x_2))_{k \geq 1}$ is bounded, and again by the Bolzano–Weierstrass Theorem, there is a subsequence $n_2(k)$ of the sequence $n_1(k)$ such that $(f_{n_2(k)}(x_2))_{k \geq 1}$ converges. Continuing this process, we construct subsequences $n_i(k)$ such that $(f_{n_i(k)}(x_i))_{k \geq 1}$ converges for all $i$. Now consider the sequence $n_k = n_k(k)$. It is a subsequence of each of the sequences $n_i(k)$. In particular, it follows that $(f_{n_k}(x_i))_{k \geq 1}$ converges for all $i$. Now we can just apply Theorem 9.12 to conclude that $f_{n_k} \overset{w^*}{\to}$ converges.

We note that an analogue of Theorem 9.13 for weakly convergent sequences is false (cf. Example 9.8).

### 9.3 Convergence of sequences of operators

Now we discuss convergence of bounded sequences of linear operators. There are three different types of convergence that arise naturally.

**Definition 9.14.** Let $X$ and $Y$ be normed spaces, and $T_n : X \to Y$ and $T : X \to Y$ are bounded linear operators. We say that:

- $T_n \overset{w}{\to} T$ if $\lim_{n \to \infty} \|T_n(x) - T(x)\| = 0$ for all $x \in X$.
- $T_n \overset{w^*}{\to} T$ if $\lim_{n \to \infty} \|T_n^*(x^*) - T^*(x^*)\| = 0$ for all $x^* \in X^*$.
- $T_n \overset{w^*}{\to} T$ if $\lim_{n \to \infty} \|T_n - T\| = 0$.

We say that $T_n \overset{w}{\to} T$ if $(T_n(x))_{n \geq 1}$ converges for all $x \in X$. We say that $T_n \overset{w^*}{\to} T$ if $(T_n^*(x^*))_{n \geq 1}$ converges for all $x^* \in X^*$.
• $T_n$ converges uniformly to $T$ (notation: $T_n \to T$) if

$$\|T_n - T\| \to 0.$$ 

• $T_n$ converges strongly to $T$ (notation: $T_n \overset{s}{\to} T$) if

$$T_n x \to T x \quad \text{for all } x \in X.$$ 

• $T_n$ converges weakly to $T$ (notation: $T_n \overset{w}{\to} T$) if

$$f(T_n x) \to f(T x) \quad \text{for all } x \in X \text{ and } f \in Y^*.$$ 

It is not hard to see that

uniform convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence

However, the converses are not true in general.

We illustrate the notions of convergence by several examples.

Example 9.15. Consider the sequence of operators $T_n : \ell^2 \to \ell^2$ defined by

$$T_n x = (x_{n+1}, x_{n+2}, \ldots), \quad x = (x_n)_{n \geq 1} \in \ell^2.$$ 

Clearly, $\|T_n x\| \leq \|x\|$, for all $x$, and for $k > n$, $\|T_n(e_k)\| = \|e_{k-n}\| = 1$. Hence, $\|T_n\| = 1$. In particular, it follows that $T_n \not\to 0$.

On the other hand, for every $x \in \ell^2$, 

$$\|T_n x\| = \sqrt{\sum_{k=n+1}^{\infty} |x_k|^2} \to 0.$$ 

Hence, $T_n \overset{s}{\to} T$.

Example 9.16. Consider the sequence of operators $T_n : \ell^2 \to \ell^2$ defined by

$$T_n x = (0, \ldots, 0, x_1, x_2, \ldots), \quad x = (x_n)_{n \geq 1} \in \ell^2.$$ 

Then $\|T_n e_1\| = \|e_1\| = 1$. Hence, $T_n \not\overset{w}{\to} T$.

On the other hand, we claim that $T_n \overset{w}{\to} T$. We recall that $(\ell^2)^* \simeq \ell^2$, and every element in $(\ell^2)^*$ is given by

$$x \mapsto \langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k.$$
for some $y = (y_n)_{n \geq 1} \in \ell^2$. For every $x, y \in \ell^2$,

$$|\langle T_n x, y \rangle| = \left| \sum_{k=1}^{\infty} x_k y_{k+n} \right| \leq \sqrt{\sum_{k=1}^{\infty} |x_k|^2} \sqrt{\sum_{k=n+1}^{\infty} |y_k|^2} \to 0.$$ 

This proves that $T_n \xrightarrow{w} T$.

**Example 9.17.** Let $X = C_c(\mathbb{R})$ be the space of continuous functions with compact support equipped with the max-norm. We consider the family of operators $T_a : X \to X$ defined by

$$T_a(\phi)(t) = \phi(t + a).$$

It is natural to expect that this family of operators depends continuously on $a$. Indeed, it follows from uniform continuity that for every $\phi \in X$,

$$\|T_a(\phi) - \phi\|_{\infty} = \max_t |\phi(t + a) - \phi(t)| \to 0$$

as $a \to \infty$. This shows that the map $a \to T_a$ is continuous at $a = 0$ with respect to the strong convergence.

On the other hand, for every fixed $a > 0$, we may consider a function $\phi \in C_c(\mathbb{R})$ such that $\{\phi \neq 0\} \subset [-a/3, a/3]$. Then it is easy to check that

$$\|T_a(\phi) - \phi\|_{\infty} = \|\phi\|_{\infty}.$$

So that $\|T_n - T_0\| = 1$, the map $a \to T_a$ is not continuous at $a = 0$ with respect to the uniform convergence.

We record some important properties of weak convergence.

**Theorem 9.18.** If $T_n \in B(X, Y)$ is a weakly convergent sequence and $X$ is a Banach space, then the sequence of norms $\|T_n\|$ is bounded.

**Proof.** If $T_n \xrightarrow{w} T$, then for every $x \in X$, $T_n x \xrightarrow{w} T x$. Then by Theorem 9.4(iii), the sequence $(T_n x)_{n \geq 1}$ is bounded for every $x \in X$. Now the claim of the theorem follows from the Uniform Boundedness Principle.

We also have an analogue of Theorem 9.5:

**Theorem 9.19.** Let $T_n, T \in B(X, Y)$ where $X$ and $Y$ are normed spaces. The sequence $(T_n)_{n \geq 1}$ converges strongly to $T$ provided that

(i) $\|T_n\|$ is uniformly bounded,

(ii) For every element $x$ in a dense subset $M \subset X$, we have $T x_n \to T x$.

This theorem is proved exactly as Theorem 9.5.