

COUNTING LATTICE POINTS

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ABSTRACT. For a locally compact second countable group G and a lattice subgroup Γ , we give an explicit quantitative solution of the lattice point counting problem in general domains in G , provided that

- i) G has finite upper local dimension, and the domains satisfy a basic regularity condition,
- ii) the mean ergodic theorem for the action of G on G/Γ holds, with a rate of convergence.

The error term we establish matches the best current result for balls in symmetric spaces of simple higher-rank Lie groups, but holds in much greater generality.

A significant advantage of the ergodic theoretic approach we use is that the solution to the lattice point counting problem is uniform over families of lattice subgroups provided they admit a uniform spectral gap. In particular, the uniformity property holds for families of finite index subgroups satisfying a quantitative variant of property τ .

We discuss a number of applications, including: counting lattice points in general domains in semisimple S -algebraic groups, counting rational points on group varieties with respect to a height function, and quantitative angular (or conical) equidistribution of lattice points in symmetric spaces and in affine symmetric varieties.

We note that the mean ergodic theorems which we establish are based on spectral methods, including the spectral transfer principle and the Kunze-Stein phenomenon. We formulate and prove appropriate analogues of both of these results in the set-up of adèle groups, and they constitute a necessary step in our proof of quantitative results in counting rational points.

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1. INTRODUCTION, DEFINITIONS, AND STATEMENTS OF GENERAL COUNTING RESULTS

1.1. Introduction and definitions. Let G be a locally compact second countable (non-compact) group and Γ a discrete subgroup of G with finite covolume. The purpose of the present paper is to give a general solution to the problem of counting lattice points in families of domains in G . More explicitly, our goal is to show that for a family of subsets $B_t \subset G$, $t > 0$,

$$(1.1) \quad |\Gamma \cap B_t| \sim m_G(B_t) \quad \text{as } t \rightarrow \infty,$$

where m_G is Haar measure on G normalised by $m_{G/\Gamma}(G/\Gamma) = 1$. Furthermore, we seek to establish an error term in the asymptotic, of the form

$$(1.2) \quad \left| \frac{|\Gamma \cap B_t|}{m_G(B_t)} - 1 \right| \leq C m_G(B_t)^{-\delta}.$$

Our approach is based on the the following fundamental principle : *the main term in the number of lattice points follows from the mean ergodic theorem in $L^2(G/\Gamma)$ for the Haar-uniform averages supported on the sets B_t , and the error estimate follows from the rate of convergence of these averages.* This principle is a part of the general ergodic theory of lattice subgroups formulated in [GN] and here we systematically develop and refine the diverse counting results which it implies.

In general, given a family of domains $B_t \subset G$ and an ergodic measure-preserving action of G on a probability measure space (X, μ) , the mean ergodic theorem (for the family B_t) is the statement that

$$(1.3) \quad \frac{1}{m_G(B_t)} \int_{B_t} f(g^{-1}x) dm_G(g) \xrightarrow{L^2} \int_X f d\mu \quad \text{as } t \rightarrow \infty$$

for every $f \in L^2(X)$. We show that the mean ergodic theorem, together with a mild regularity property for the sets B_t (namely, well-roundedness [DRS], [EM]), implies that (1.1) holds. Furthermore, when convergence takes place with a fixed rate, the sets B_t satisfy a quantitative regularity condition (namely, Hölder well-roundedness [GN]), and G has finite upper local dimension, then the lattice point counting problem for the domains B_t admits an explicit quantitative solution. The error term is controlled directly by the spectral gap estimate satisfied by the family of averaging operators above acting on $L^2(G/\Gamma)$, together with the degree of regularity of B_t and the upper local dimension.

Previous counting results in the literature are improved upon in several different respects, including admitting more general sets, establishing or improving explicit error terms, and enlarging the class of groups involved. Our approach also gives uniform estimates over families of lattice subgroups (as well as over their cosets), which have a number of interesting applications (see Section 1.4).

We begin by recalling and introducing some definitions needed in the statements of the main results. Let \mathcal{O}_ε , $\varepsilon > 0$, be a family of symmetric neighbourhoods of the identity in G , which is decreasing with ε . Let $B_t \subset G$, $t \in \mathbb{R}_+$, be a family of bounded Borel subsets of positive finite Haar measure. In the following definition, we recall the notion of well-rounded sets from [DRS] and [EM], and give an effective version of it (see [GN]).

Definition 1.1. Well-rounded and Hölder well-rounded sets.

- (1) The family B_t is *well-rounded* (w.r.t. \mathcal{O}_ε) if for every $\delta > 0$ there exist $\varepsilon, t_1 > 0$ such that for all $t \geq t_1$,

$$m_G(\mathcal{O}_\varepsilon B_t \mathcal{O}_\varepsilon) \leq (1 + \delta) m_G(\cap_{u,v \in \mathcal{O}_\varepsilon} u B_t v).$$

- (2) The family B_t is *Hölder well-rounded* with exponent a (w.r.t. \mathcal{O}_ε) if there exist $c, \varepsilon_1, t_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$ and $t \geq t_1$,

$$m_G(\mathcal{O}_\varepsilon B_t \mathcal{O}_\varepsilon) \leq (1 + c\varepsilon^a) m_G(\cap_{u,v \in \mathcal{O}_\varepsilon} u B_t v).$$

Given a family B_t of subsets of G , we set

$$(1.4) \quad B_t^+(\varepsilon) = \mathcal{O}_\varepsilon B_t \mathcal{O}_\varepsilon \quad \text{and} \quad B_t^-(\varepsilon) = \cap_{u,v \in \mathcal{O}_\varepsilon} u B_t v.$$

Let us also recall the following natural condition, which is clearly stronger than Hölder well-roundedness.

Definition 1.2. Admissible 1-parameter families [GN]. The family B_t is *Hölder admissible* with exponent a (w.r.t. \mathcal{O}_ε) if there exist $c, t_1, \varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$ and $t \geq t_1$,

$$\begin{aligned} \mathcal{O}_\varepsilon \cdot B_t \cdot \mathcal{O}_\varepsilon &\subset B_{t+c\varepsilon^a}, \\ m_G(B_{t+\varepsilon}) &\leq (1 + c\varepsilon^a) \cdot m_G(B_t). \end{aligned}$$

Hölder admissibility (and in some considerations even Lipschitz admissibility) played an important role in the arguments in [GN] applied to prove pointwise ergodic theorems for general actions of a group G and a lattice subgroup Γ . For an extensive list of example of admissible averages on S -algebraic groups we refer to [GN, Ch. 7]. However, as already noted in [GN], when we consider only the mean ergodic theorem on spaces of the form G/Γ , the condition of Hölder well-roundedness will be sufficient. This condition allows for a very diverse set of averages, as we shall see in the examples below. For instance, the sets arising in the study of angular distribution of lattice points (see Sections 7 and 8) are Hölder well-rounded, but not Hölder admissible.

The family of neighbourhoods \mathcal{O}_ε gives rise to the notion of upper local dimension:

Definition 1.3. Upper Local dimension. We say that the *upper local dimension* is at most ρ if there exist $m_0, \varepsilon_1 > 0$ such that

$$m_G(\mathcal{O}_\varepsilon) \geq m_0 \varepsilon^\rho \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

For example, when G is a connected Lie group, we fix a Riemannian metric on G and set $\mathcal{O}_\varepsilon = \{g \in G : d(g, e) < \varepsilon\}$. Then one can take $\rho = \dim G$. Another important example is the case where $G = G_\infty \times G_f$ is a product of a connected Lie group of positive dimension, and a totally disconnected group G_f . Set $\mathcal{O}_\varepsilon = \mathcal{O}_\varepsilon^\infty \times W$, where $\mathcal{O}_\varepsilon^\infty$ are Riemannian balls of radius ε centered at the identity of G_∞ , and W is a fixed compact open subgroup of G_f . Then again ρ is the dimension of G_∞ .

Let β_t denote the normalised Haar-uniform measure supported on the set B_t . Consider a measure-preserving action of G on a standard Borel probability space (X, μ) and the averaging operators $\pi_X(\beta_t)$ defined by

$$(1.5) \quad \pi_X(\beta_t)f(x) := \frac{1}{m_G(B_t)} \int_{B_t} f(g^{-1}x) dm_G(g), \quad f \in L^p(X).$$

Definition 1.4. Mean ergodic theorems.

- (1) The operators $\pi_X(\beta_t)$ satisfy the *mean ergodic theorem in $L^2(X)$* if

$$\left\| \pi_X(\beta_t)f - \int_X f d\mu \right\|_{L^2(X)} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for all $f \in L^2(X)$.

- (2) The operators $\pi_X(\beta_t)$ satisfy the *quantitative mean ergodic theorem in $L^2(X)$* with rate $E(t)$ if

$$\left\| \pi_X(\beta_t)f - \int_X f d\mu \right\|_{L^2(X)} \leq E(t) \|f\|_{L^2(X)}$$

for all $f \in L^2(X)$ and $t > 0$, where $E : (0, \infty) \rightarrow (0, \infty)$ is a function such that $E(t) \rightarrow 0$ as $t \rightarrow \infty$.

Note that when the action of G on X satisfies the quantitative mean ergodic theorem, the unitary representation of G in $L^2_0(X)$ (the space of mean zero functions) must have a spectral gap, provided that at least one of the sets $B_t^{-1}B_t$ generates G for some t such that $E(t) < 1$. Conversely, when G is a connected semisimple Lie group with finite center and the unitary representation of G in $L^2_0(X)$ has a (strong) spectral gap, for general families B_t one has a quantitative mean ergodic theorem of the form

$$\left\| \pi_X(\beta_t)f - \int_X f d\mu \right\|_{L^2(X)} \leq C m(B_t)^{-\kappa} \|f\|_{L^2(X)}$$

for some $C, \kappa > 0$ (see Theorem 4.5 below).

In order to solve the lattice point counting problem, we will need a stable version of the mean ergodic theorem:

Definition 1.5. Stable mean ergodic theorems. We will call the ergodic theorem for averages along the sets B_t *stable* if it holds for all the families $B_t^+(\varepsilon)$ and $B_t^-(\varepsilon)$ simultaneously for $\varepsilon \in (0, \varepsilon_1)$. For the quantitative mean ergodic theorem we require in addition that the function $E(t)$ is independent of ε .

Remark 1.6. As we shall see, in the context of semisimple S -algebraic groups a well-rounded family which satisfies the (quantitative) mean ergodic theorem also satisfies the (quantitative) stable mean ergodic theorem. Indeed, our method of establishing the norm estimate associated with a strong spectral gap is based on the spectral transfer principle and the Kunze-Stein phenomenon. Together these imply that the rate of convergence depends only on the rate of volume growth of the family, and the L^p -parameter of integrability of the representation. We will establish and then apply similar considerations to radial averages on adèle groups.

1.2. Statement of general counting results. We can now formulate the following basic result, which provides the main term in the lattice point counting problem in well-rounded domains. Fixing a choice of Haar measure on G , let us denote the measure of a fundamental domain of Γ in G by $V(\Gamma)$.

Theorem 1.7. Lattice point problem : main term. *Let G be an lsc group, $\Gamma \subset G$ a discrete lattice subgroup, and B_t a well-rounded family of subsets of G . Assume that the averages β_t supported on B_t satisfy the stable mean ergodic theorem in $L^2(G/\Gamma)$. Then*

$$\lim_{t \rightarrow \infty} \frac{|\Gamma \cap B_t|}{m_G(B_t)} = \frac{1}{V(\Gamma)}.$$

The argument of the proof of Theorem 1.7 also applies to count points in translated cosets of lattice subgroups.

Corollary 1.8. *Under the conditions of Theorem 1.7,*

$$\lim_{t \rightarrow \infty} \frac{|x\Gamma y^{-1} \cap B_t|}{m_G(B_t)} = \frac{1}{V(\Gamma)}$$

for every $x, y \in G$.

As noted in §1.1, to handle the error term we will use a quantitative estimate on the rate of L^2 -norm convergence of the averages $\pi_{G/\Gamma}(\beta_t)$ to the ergodic mean, together with a quantitative form of well-roundedness. This will give a uniform quantitative solution to the lattice point counting problem. Before we formulate this result, we summarise our notation:

$$(1.6) \quad \begin{aligned} c, a &= \text{the Hölder well-roundedness parameters of the family } B_t, \\ m_0, \rho &= \text{the local upper dimension estimate for the group } G, \\ E(t) &= \text{the error estimate in the stable mean ergodic theorem for } B_t. \end{aligned}$$

For $g \in G$, we set

$$\varepsilon_0(g, \Gamma) = \sup\{\varepsilon > 0 : \mathcal{O}_\varepsilon^2 g \text{ injects in } G/\Gamma\}.$$

Theorem 1.9. Lattice point problem : error term. *Let G be an lsc group, and B_t a Hölder well-rounded family of subsets of G , w.r.t. a family \mathcal{O}_ε of upper local dimension at most ρ . Let $\Gamma \subset G$ be any discrete lattice subgroup, and assume that the averages β_t satisfy the stable quantitative mean ergodic theorem in $L^2(G/\Gamma)$ with rate $E(t)$, and that $\varepsilon_0(e, \Gamma) \geq \varepsilon_0$. Then there exists $t_0 > 0$ such that for $t \geq t_0$,*

$$(1.7) \quad \left| \frac{|\Gamma \cap B_t|}{m_G(B_t)} - \frac{1}{V(\Gamma)} \right| \leq A E(t)^{a/(\rho+a)},$$

where $A = (4m_0^{-1})^{a/(\rho+a)} (c m_G(\mathcal{O}_{\varepsilon_0})^{-1})^{\rho/(\rho+a)}$.

Remark 1.10. Let us note the following regarding Theorem 1.9.

- (1) The estimate (1.7) is independent of the choice of Haar measure m_G .
- (2) If the sets B_t are bi-invariant under a compact subgroup K of G , we can take ρ to be the upper local dimension of $K \backslash G$. Indeed, then the proof of Theorem 1.9 can then be carried out in the space $L^2(K \backslash G/\Gamma)$.
- (3) The constant t_0 depends on all the parameters in (1.6), as well as $\varepsilon_0, \varepsilon_1$, and t_1 (appearing in Definitions 1.2, 1.3, 1.5).

As to counting points in translated cosets of the lattice, we note the following result which will be shown below to follow from Theorem 1.9.

Corollary 1.11. *Under the conditions of Theorem 1.9, there exists $t_0 > 0$ such that for $x, y \in G$, satisfying $\varepsilon_0(x, \Gamma), \varepsilon_0(y, \Gamma) \geq \varepsilon_0$, and $t \geq t_0$,*

$$(1.8) \quad \left| \frac{|x\Gamma y^{-1} \cap B_t|}{m_G(B_t)} - \frac{1}{V(\Gamma)} \right| \leq A E(t)^{a/(\rho+a)}$$

where A is the same as in Theorem 1.9.

Remark 1.12. If Ω is a subset of G bounded modulo Γ , then (1.8) holds for all $x, y \in \Omega$ and $t \geq t_0(\text{diam}(\Omega\Gamma))$, where the constant A is given as in Theorem 1.9 with $\varepsilon_0 = \inf\{\varepsilon(g, \Gamma) : g \in \Omega\}$. In particular, if the lattice Γ is co-compact, (1.8) holds uniformly over all $x, y \in G$.

An important consequence of Theorem 1.9 is that given the family B_t , the error estimate depends only on the size of the spectral gap, and on the size of the largest neighbourhood $\mathcal{O}_\varepsilon^2$ which injects into G/Γ . In particular, the error estimate holds *uniformly* for all lattice subgroups in G for which these two parameters have fixed lower bounds.

Specialising Theorem 1.9 further, we fix the lattice Γ , and note that then the error estimate holds uniformly over an infinite family of finite index subgroups of Γ , provided only that the family satisfies a uniform spectral estimate (which is a quantitative version of property τ). This fact will be exploited in Section 5, where we consider the congruence subgroups of an arithmetic lattice, and has other uses as well. We formulate it separately, as follows.

Corollary 1.13. *Assume that the conditions of Theorem 1.9 hold for the lattice Γ_0 and a family Γ_j , $j \in \mathbb{N}$ of its finite-index subgroups. Given a set Ω which is compact modulo Γ_0 , there exists $t_0 > 0$ such that for $t \geq t_0$ and $x, y \in \Omega$, uniformly for all $j \in \mathbb{N}$*

$$\left| \frac{|x\Gamma_j y^{-1} \cap B_t|}{m_G(B_t)} - \frac{1}{V(\Gamma_j)} \right| \leq A E(t)^{a/(\rho+a)}$$

where A is given as in Theorem 1.9 with $\varepsilon_0 = \inf\{\varepsilon_0(g, \Gamma) : g \in \Omega\}$.

1.3. Comparison with the existing literature. The problem of the asymptotic development of the number of lattice points in Euclidean space has a long history, going back to Gauss, who initiated the study of the number of integral points in domains of the Euclidean plane. For Euclidean space, and more generally for spaces with polynomial volume growth, there is a simple geometric argument to derive the main term in the asymptotic, and an error estimate for sufficiently regular convex sets has been established using the Fourier transform [Hl][Hz]. For Lie groups of exponential volume growth already the main term, and certainly the error term, require significant analytic techniques. The first non-Euclidean counting result in the semisimple case is due to Delsarte [D]. Currently there are several different approaches to non-Euclidean lattice point counting problems for $\Gamma \subset G$ in certain cases, as follows.

- (1) via direct spectral expansion and regularisation of the automorphic kernel on G/Γ , G semisimple [BT1, BT2, BMW, CLT, D, DRS, Go, Hu, LP, Le, Pa, MW, ST, STBT1, STBT2, Se],
- (2) via mixing of 1-parameter flows, or equivalently decay of matrix coefficients for semisimple groups [Bar1, Bar2, BO, EM, GMO, Mar1, Mar2, Mar3, Mau, Mu1, Mu2],
- (3) via symbolic coding of Anosov flows and transfer operator techniques [La, Po, Q, Sh], for lattices in simple groups of real rank one,
- (4) via the Weil bound for Kloosterman sums [Boc, HZ], when counting in congruence subgroups of SL_2 ,
- (5) via equidistribution of unipotent flows [EMS, GO1, GO2] (this approach does not provide an error term in the asymptotic).

Our approach, which is different from those listed above, is based on the mean ergodic theorem for the averages β_t acting on G/Γ , and has a number of advantages:

- *Simplicity of the method.* The quantitative mean ergodic theorem as well as the resulting estimates for counting lattice points are established by relatively elementary spectral and geometric comparison arguments which hold in great generality. They are valid, in particular, for all semisimple S -algebraic groups and semisimple adèle groups, but are not restricted to them. The arguments avoid the complications arising from direct spectral expansion of the automorphic kernel on G/Γ (in particular, those associated with regularisation of Eisenstein series) which method (1) introduces. As a result, this approach allows a considerable expansion of the scope of quantitative lattice point counting results.
- *Quality of error term.* The quality of the error term derived via the quantitative mean ergodic theorem matches or exceeds the currently known best bound in all the non-Euclidean lattice point counting problems we are aware of, with the exception of [Se], [LP], [Pa] and [BMW]. Note that the latter results deal only with lattices in real rank one Lie groups (or their products), and only with sets which are bi-invariant under a maximal compact subgroup. These assumptions make it possible to deploy method (1) using very detailed information regarding the special functions appearing in the spherical spectral expansion. For more general domains, the approach via the decay of matrix coefficients in method (2) can be used to give an error estimate in the semisimple case, but its quality is inferior to the one stated above. (For more details cf. [GN, Ch. 2] and compare with [Mau] for the case of real groups, and with [GMO] for the case of adèle groups).
- *Uniformity over lattice families.* The quantitative ergodic theorem governing the behaviour of the averages β_t is valid for all ergodic probability-measure-preserving actions of G satisfying the same spectral bound. In particular it holds uniformly for all the homogeneous probability spaces G/Γ , as Γ ranges over lattice subgroups of G , provided they satisfy a fixed spectral gap estimate. Together with an obvious necessary geometric condition on the fundamental domains, our approach solves the lattice point counting problem uniformly over this class of lattices. In particular, the counting result holds uniformly for all the finite index subgroups of an irreducible lattice Γ in a semisimple S -algebraic group, which satisfy a uniform spectral gap property, namely property τ , for example congruence subgroups (see §5 and §1.4 for some applications of the uniformity property).
- *Generality of the sets.* The mean ergodic theorem is typically very robust, namely it holds in great generality and is usually stable under geometric perturbations of the averaging sets. The same two features hold for the quantitative mean ergodic theorem on the homogeneous probability space G/Γ . This allows us to establish the first quantitative counting results for general families of non-radial sets on semisimple groups, including such natural examples as bi-sectors on symmetric spaces and affine symmetric spaces (see Section 7 and Section 8). Note for example that our error estimate for tempered lattices in sectors in the hyperbolic plane matches the one recently obtained in [Boc] via method (4), but applies for all lattices (in all semisimple groups) rather than just congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

- *Generality of the groups.* Mean ergodic theorems originated in amenable ergodic theory and of course do not require the group to be semisimple. The lattice point counting results we present are valid in the case of lattices in amenable groups as well, for example in connected nilpotent or exponential-solvable Lie groups. Other cases where they can be applied are the affine groups of Euclidean spaces, as well as the associated adèle groups. We will treat these matters in more detail elsewhere, but emphasize here that the principle of deriving the solution to the lattice point counting problem from the mean ergodic theorem is completely general and valid for every lsc group. In particular, the assumption of mixing for flows on the space G/Γ — called for in method (2) — is not relevant to the lattice point counting problem.

1.4. Applications of uniformity in counting. Theorem 1.9 and its corollaries have already found a number of applications beyond those we will describe below in the present paper. In addition to numerous examples discussed in [GN, Ch. 2], let us mention very briefly the following consequences of the uniformity in counting over congruence subgroups of an arithmetic lattice in an algebraic group.

- (1) For connected semisimple Lie groups, uniformity was first stated and utilised in the problem of sifting the integral points $G(\mathbb{Z})$ on the group variety $G(\mathbb{R})$ [NS]. It plays an essential role in establishing the existence of the right order of magnitude of almost prime points, for example, almost prime integral unimodular matrices. Theorem 5.1 which we formulate below allows the generalisation of these results to S -algebraic groups, for example to construct almost prime unimodular S -integral matrices. It is also crucial in establishing the existence of almost prime points on affine symmetric varieties (for example, integral symmetric matrices).
- (2) Given a affine homogeneous variety X of a semisimple algebraic group defined over \mathbb{Q} , it is possible to establish using uniformity in counting over congruence groups, an effective result on lifting of integral points. Namely, we show that every point in the image of the reduction map $X(\mathbb{Z}) \rightarrow X(\mathbb{Z}/p\mathbb{Z})$ can be lifted to a point with coefficients of size $O(p^N)$ with fixed $N > 0$. On the other hand, such a result is false for general homogeneous varieties.
- (3) We obtain an estimate on the number of integral points on certain proper subvarieties X of a group variety G . Namely, we show that there exists uniform $\alpha \in (0, 1)$ such that $N_T(X) = O_X(N_T(G)^\alpha)$, where $N_T(\cdot)$ denotes the number of integral points with norm bounded by T .

The proofs of these results will be given in a separate paper.

1.5. Organisation of the paper. We prove the results stated in §1.2 in Section 2. In Section 3 we formulate a general recipe for counting lattice points and explain the case of sectors in the hyperbolic plane as a motivating example. In the rest of the paper, we discuss several applications of our results. We discuss lattices in semisimple S -algebraic groups (Section 4), uniformity over congruence subgroups and the density hypothesis (Section 5), rational points on group varieties (Section 6), and angular distribution of lattice points on symmetric spaces (Section 7) and on affine symmetric varieties (Section 8).

2. ERGODIC THEOREMS AND COUNTING LATTICE POINTS

In this section, we prove the results stated in the introduction. Let G be an lsc group and Γ a discrete lattice subgroup of G . We denote by $\tilde{m}_{G/\Gamma}$ the measure induced on G/Γ by our fixed choice of the Haar measure m_G on G . Thus, $\tilde{m}_{G/\Gamma}(G/\Gamma)$ is the total measure of the locally symmetric space G/Γ , a quantity we denote by $V(\Gamma)$. We also let $m_{G/\Gamma}$ denote the corresponding probability measure on G/Γ , namely $\tilde{m}_{G/\Gamma}/V(\Gamma)$. Let \mathcal{O}_ε be a family of symmetric neighbourhoods of the identity and

$$\chi_\varepsilon = \frac{\chi_{\mathcal{O}_\varepsilon}}{m_G(\mathcal{O}_\varepsilon)}.$$

We consider the function:

$$\phi_\varepsilon(g\Gamma) = \sum_{\gamma \in \Gamma} \chi_\varepsilon(g\gamma).$$

Note that ϕ_ε is a measurable bounded function on G/Γ with compact support, and

$$(2.1) \quad \int_G \chi_\varepsilon dm_G = 1, \quad \int_{G/\Gamma} \phi_\varepsilon d\tilde{m}_{G/\Gamma} = 1, \quad \int_{G/\Gamma} \phi_\varepsilon dm_{G/\Gamma} = \frac{1}{V(\Gamma)}.$$

Let us now note the following basic observations, which will allow us to reduce the lattice point counting problem to the ergodic theorem on G/Γ , together with a regularity property of the domains.

First, for any $\delta > 0$, $h \in G$ and $t > 0$, the following are obviously equivalent, by definition, for any family of Haar-uniform measures b_t on G ,

$$(2.2) \quad \left| \pi_{G/\Gamma}(b_t)\phi_\varepsilon(h\Gamma) - \frac{1}{V(\Gamma)} \right| \leq \delta,$$

$$(2.3) \quad \frac{1}{V(\Gamma)} - \delta \leq \frac{1}{m_G(\text{supp } b_t)} \int_{\text{supp } b_t} \phi_\varepsilon(g^{-1}h\Gamma) dm_G(g) \leq \frac{1}{V(\Gamma)} + \delta.$$

We will estimate the first expression using the mean ergodic theorem and Chebycheff's inequality. On the other hand, the integral in the second expression is connected to lattice points as follows.

Lemma 2.1. *Let B_t be a family of measurable subsets of G . Then for every $t > 0$, $\varepsilon > 0$ and $h \in \mathcal{O}_\varepsilon$,*

$$\int_{B_t^-(\varepsilon)} \phi_\varepsilon(g^{-1}h\Gamma) dm_G(g) \leq |B_t \cap \Gamma| \leq \int_{B_t^+(\varepsilon)} \phi_\varepsilon(g^{-1}h\Gamma) dm_G(g)$$

where $B_t^+(\varepsilon)$ and $B_t^-(\varepsilon)$ are defined as in (1.4).

Proof. If $\chi_\varepsilon(g^{-1}h\gamma) \neq 0$ for some $g \in B_t^-(\varepsilon)$, $h \in \mathcal{O}_\varepsilon$, $\gamma \in \Gamma$, then we obtain

$$\gamma \in h^{-1} \cdot B_t^-(\varepsilon) \cdot (\text{supp } \chi_\varepsilon) \subset B_t$$

since $\mathcal{O}_\varepsilon B_t^-(\varepsilon) \mathcal{O}_\varepsilon \subset B_t$. Hence, by the definition of ϕ_ε and (2.1),

$$\int_{B_t^-(\varepsilon)} \phi_\varepsilon(g^{-1}h\Gamma) dm_G(g) = \sum_{\gamma \in B_t \cap \Gamma} \int_{B_t} \chi_\varepsilon(g^{-1}h\gamma) dm_G(g) \leq |B_t \cap \Gamma|.$$

In the other direction, for $\gamma \in B_t \cap \Gamma$ and $h \in \mathcal{O}_\varepsilon$,

$$\text{supp}(g \mapsto \chi_\varepsilon(g^{-1}h\gamma)) = h\gamma(\text{supp } \chi_\varepsilon)^{-1} \subset B_t^+(\varepsilon).$$

Since $\chi_\varepsilon \geq 0$ and (2.1) holds,

$$\int_{B_t^+(\varepsilon)} \phi_\varepsilon(g^{-1}h\Gamma) dm_G(g) \geq \sum_{\gamma \in B_t \cap \Gamma} \int_{B_t^+(\varepsilon)} \chi_\varepsilon(g^{-1}h\gamma) dm_G(g) = |B_t \cap \Gamma|,$$

as required. \square

Proof of Theorem 1.7. We use a small parameters $\delta > 0$. Since the family B_t is well-rounded, there exists $\varepsilon > 0$ such that

$$(2.4) \quad m_G(B_t^+(\varepsilon)) \leq (1 + \delta)m_G(B_t^-(\varepsilon))$$

for all sufficiently large t .

By the stable mean ergodic theorem and (2.1),

$$\left\| \frac{1}{m_G(B_t^+(\varepsilon))} \int_{B_t^+(\varepsilon)} \phi_\varepsilon(g^{-1}h\Gamma) dm_G(g) - \frac{1}{V(\Gamma)} \right\|_{L^2(G/\Gamma)} \rightarrow 0$$

and so

$$m_{G/\Gamma} \left(\left\{ h\Gamma : \left| \frac{1}{m_G(B_t^+(\varepsilon))} \int_{B_t^+(\varepsilon)} \phi_\varepsilon(g^{-1}h\Gamma) dm_G(g) - \frac{1}{V(\Gamma)} \right| > \delta \right\} \right) \rightarrow 0$$

as $t \rightarrow \infty$. Hence, the measure of this set will be less than $m_G(\mathcal{O}_\varepsilon\Gamma)$ for large t . Then there exists $h_t \in \mathcal{O}_\varepsilon$ such that

$$\left| \frac{1}{m_G(B_t^+(\varepsilon))} \int_{B_t^+(\varepsilon)} \phi_\varepsilon(g^{-1}h_t\Gamma) dm_G(g) - \frac{1}{V(\Gamma)} \right| \leq \delta.$$

Combining this estimate with Lemma 2.1 and (2.4), we obtain that

$$|\Gamma \cap B_t| \leq \left(\frac{1}{V(\Gamma)} + \delta \right) m_G(B_t^+(\varepsilon)) \leq \left(\frac{1}{V(\Gamma)} + \delta \right) (1 + \delta)m_G(B_t)$$

for all $\delta > 0$ and $t \geq t_1(\delta)$. Since one can similarly prove the lower estimate, this completes the proof. \square

Proof of Corollary 1.8. We apply both parts of the proof of Theorem 1.7 to the function in $L^2(G/\Gamma)$ given by

$$(2.5) \quad \phi_\varepsilon^y(gx\Gamma) = \sum_{\gamma \in \Gamma} \chi_\varepsilon(gx\gamma y^{-1}).$$

\square

Proof of Theorem 1.9. Recall that we are now assuming that the family $\{B_t\}$ is Hölder well-rounded:

$$m_G(B_t^+(\varepsilon)) \leq (1 + c\varepsilon^a)m_G(B_t^-(\varepsilon)) \quad \text{for every } t > t_1 \text{ and } \varepsilon \in (0, \varepsilon_1),$$

the neighbourhoods \mathcal{O}_ε satisfy

$$(2.6) \quad m_G(\mathcal{O}_\varepsilon) \geq m_0\varepsilon^\rho \quad \text{for some } m_0 > 0 \text{ and every } \varepsilon \in (0, \varepsilon_1),$$

and the averages along the sets $B_t^\pm(\varepsilon)$, $0 < \varepsilon < \varepsilon_1$, satisfy the stable quantitative mean ergodic theorem with the error term $E(t)$.

In the proof, we use a parameter $\varepsilon > 0$ satisfying

$$(2.7) \quad \varepsilon < \min\{\varepsilon_0, \varepsilon_1, c^{-1/a}\}$$

where ε_0 is such that the projection $\mathcal{O}_{\varepsilon_0}^2 \rightarrow \mathcal{O}_{\varepsilon_0}^2 \Gamma$ is injective. Since the neighbourhoods \mathcal{O}_ε are symmetric,

$$(2.8) \quad \mathcal{O}_{\varepsilon_0} \gamma_1 \cap \mathcal{O}_{\varepsilon_0} \gamma_2 = \emptyset \quad \text{for } \gamma_1 \neq \gamma_2 \in \Gamma,$$

and

$$(2.9) \quad m_{G/\Gamma}(\mathcal{O}_\varepsilon \Gamma) = \frac{m_G(\mathcal{O}_\varepsilon)}{V(\Gamma)}.$$

Also, we have

$$(2.10) \quad V(\Gamma) \geq m(\mathcal{O}_{\varepsilon_0}).$$

For any family of Haar-uniform averages b_t satisfying the quantitative mean ergodic theorem with the error term $E(t)$ for its action on the probability space $(G/\Gamma, m_{G/\Gamma})$, we have for all $t > 0$,

$$\left\| \pi_{G/\Gamma}(b_t) \phi_\varepsilon - \int_{G/\Gamma} \phi_\varepsilon dm_{G/\Gamma} \right\|_{L^2(G/\Gamma)} \leq E(t) \|\phi_\varepsilon\|_{L^2(G/\Gamma)},$$

and thus for all $\delta, t > 0$,

$$m_{G/\Gamma} \left(\left\{ h\Gamma : \left| \pi_{G/\Gamma}(b_t) \phi_\varepsilon(h\Gamma) - \frac{1}{V(\Gamma)} \right| > \delta \right\} \right) \leq \delta^{-2} E(t)^2 \|\phi_\varepsilon\|_{L^2(G/\Gamma)}^2.$$

It follows from (2.8) that

$$\begin{aligned} \|\phi_\varepsilon\|_{L^2(G/\Gamma)}^2 &= \int_{G/\Gamma} \phi_\varepsilon(h\Gamma)^2 \frac{d\tilde{m}_{G/\Gamma}(h\Gamma)}{V(\Gamma)} \\ &= \int_G \chi_\varepsilon^2(g) \frac{dm_G(g)}{V(\Gamma)} = \frac{m_G(\mathcal{O}_\varepsilon)^{-1}}{V(\Gamma)}. \end{aligned}$$

Hence,

$$(2.11) \quad m_{G/\Gamma} \left(\left\{ h\Gamma : \left| \pi_{G/\Gamma}(b_t) \phi_\varepsilon(h\Gamma) - \frac{1}{V(\Gamma)} \right| > \delta \right\} \right) \leq \frac{m_G(\mathcal{O}_\varepsilon)^{-1}}{V(\Gamma)} \delta^{-2} E(t)^2.$$

This shows that the measure of the latter set decays with t . In particular, the measure will eventually be strictly smaller than $m_{G/\Gamma}(\mathcal{O}_\varepsilon \Gamma) = m_G(\mathcal{O}_\varepsilon)/V(\Gamma)$ for sufficiently large t . Then

$$(2.12) \quad \mathcal{O}_\varepsilon \Gamma \cap \left\{ h\Gamma : \left| \pi_{G/\Gamma}(b_t) \phi_\varepsilon(h\Gamma) - \frac{1}{V(\Gamma)} \right| \leq \delta \right\} \neq \emptyset.$$

Thus according to (2.2) and (2.3) applied to the sets $B_t^+(\varepsilon)$, for any h in the non-empty intersection (2.12),

$$(2.13) \quad \frac{1}{m_G(B_t^+(\varepsilon))} \int_{B_t^+(\varepsilon)} \phi_\varepsilon(g^{-1}h\Gamma) dm_G(g) \leq \frac{1}{V(\Gamma)} + \delta.$$

On the other hand, by Lemma 2.1, for $h \in \mathcal{O}_\varepsilon$,

$$(2.14) \quad |\Gamma \cap B_t| \leq \int_{B_t^+(\varepsilon)} \phi_\varepsilon(g^{-1}h\Gamma) dm_G(g).$$

Combining these estimates and using the fact that the family $\{B_t\}$ is Hölder well-rounded, we conclude that

$$(2.15) \quad |\Gamma \cap B_t| \leq \left(\frac{1}{V(\Gamma)} + \delta \right) m_G(B_t^+(\varepsilon)) \leq \left(\frac{1}{V(\Gamma)} + \delta \right) (1 + c\varepsilon^a) m_G(B_t).$$

This inequality holds as soon as (2.12) holds, and so certainly if we have

$$(2.16) \quad \frac{m_G(\mathcal{O}_\varepsilon)^{-1}}{V(\Gamma)} \delta^{-2} E(t)^2 \leq \frac{1}{4} \cdot \frac{m_G(\mathcal{O}_\varepsilon)}{V(\Gamma)}.$$

Indeed, then the right hand side is strictly smaller than $m_{G/\Gamma}(\mathcal{O}_\varepsilon\Gamma)$, so that the intersection (2.12) is necessarily non-empty. We set $\delta = 2m_G(\mathcal{O}_\varepsilon)^{-1}E(t)$ so that the equality in (2.16) holds. Now using (4.10), the estimate $c\varepsilon^a < 1$, (4.13), and (2.6), we deduce that

$$\begin{aligned} \frac{|\Gamma \cap B_t|}{m_G(B_t)} - \frac{1}{V(\Gamma)} &\leq 2\delta + \frac{c\varepsilon^a}{V(\Gamma)} \leq 4m_G(\mathcal{O}_\varepsilon)^{-1}E(t) + \frac{c\varepsilon^a}{m_G(\mathcal{O}_{\varepsilon_0})} \\ &\leq 4m_0^{-1}\varepsilon^{-\rho}E(t) + \frac{c\varepsilon^a}{m_G(\mathcal{O}_{\varepsilon_0})}. \end{aligned}$$

To optimise the error term, we choose

$$\varepsilon = (4m_0^{-1}c^{-1}m_G(\mathcal{O}_{\varepsilon_0})E(t))^{1/(\rho+a)}.$$

Note that since $E(t) \rightarrow 0$ as $t \rightarrow \infty$, there exists $t_0 > 0$ such that ε satisfies (2.7) for all $t \geq t_0$. Finally, we obtain that for $t \geq t_0$,

$$\frac{|\Gamma \cap B_t|}{m_G(B_t)} - \frac{1}{V(\Gamma)} \leq A E(t)^{a/(\rho+a)}$$

where $A = (4m_0^{-1})^{a/(\rho+a)}(cm_G(\mathcal{O}_{\varepsilon_0})^{-1})^{\rho/(\rho+a)}$.

Note that both the comparison argument in Lemma 2.1, as well the estimate (2.3) derived from the mean ergodic theorem, give a lower bound in addition to the foregoing upper bound. Thus the same arguments can be repeated to yield also a lower bound for the lattice points count. This completes the proof of Theorem 1.9. \square

Proof of Corollary 1.11. As in Lemma 2.1, the quantity $|B_t \cap x\Gamma y^{-1}|$ can be estimated by integrating the function ϕ_ε^y , defined in (2.5), on small perturbations of B_t . Indeed, for $\varepsilon < \varepsilon_0(y, \Gamma)$ we have

$$\mathcal{O}_\varepsilon y \gamma_1 \cap \mathcal{O}_\varepsilon y \gamma_2 = \emptyset \quad \text{for } \gamma_1 \neq \gamma_2 \in \Gamma.$$

Then the supports of the functions $g \mapsto \chi_\varepsilon(gx\gamma y^{-1})$, $\gamma \in \Gamma$, are disjoint, and we deduce that

$$\|\phi_\varepsilon^y\|_{L^2(G/\Gamma)}^2 = \frac{m_G(\mathcal{O}_\varepsilon)^{-1}}{V(\Gamma)}$$

as before. Also,

$$m_{G/\Gamma}(\mathcal{O}_\varepsilon x\Gamma) = \frac{m_G(\mathcal{O}_\varepsilon)}{V(\Gamma)} \quad \text{and} \quad V(\Gamma) \geq m_G(\mathcal{O}_{\varepsilon_0}).$$

Using this estimates, the proof proceeds exactly as in Theorem 1.9. \square

Proof of Corollary 1.13. Clearly, when all Γ_j are subgroups of a fixed lattice Γ_0 , if $\mathcal{O}_{\varepsilon_0}^2$ injects into G/Γ_0 , it also injects into G/Γ_j . Since we assume that the operators $\pi_{G/\Gamma_j}(\beta_t)$ satisfy the stable quantitative mean ergodic theorem, with the same rate $E(t)$ for all j , the result follows. \square

3. LATTICE POINT COUNTING PROBLEMS : GENERAL RECIPE AND AN EXAMPLE

3.1. General recipe. In the following sections we will give several applications of the general lattice point counting result, namely Theorem 1.9. These applications are based on the following recipe: if G is an lsc group, Γ a discrete lattice in G , and B_t a family of sets for which we wish to find the asymptotic of the number of lattice points together with an error term, Theorem 1.9 reduces the problem to the following two steps:

- (1) Establish that $\left\| \pi_{G/\Gamma}(\beta_t)f - \int_{G/\Gamma} f dm_{G/\Gamma} \right\|_2 \leq E(t) \|f\|_2$ for some decaying function $E(t)$, where β_t denotes the Haar-uniform averages supported on B_t (or “small perturbations” thereof).
- (2) Establish that the family of sets B_t is Hölder well-rounded w.r.t. to a local neighbourhood family \mathcal{O}_ε which has finite upper local dimension.

The first step is of spectral nature and requires some information regarding the unitary representation theory of G , and more specifically, the spectrum of $L^2(G/\Gamma)$. The second step is geometric, and involves the structure of a neighbourhood family \mathcal{O}_ε in G and the regularity of the sets B_t under small perturbations.

In order to carry out the first step, it will be convenient to use the notion of an L^p -representation.

Definition 3.1. L^p -representation. A unitary representation $\pi : G \rightarrow U(\mathcal{H})$ of an lsc group G is called L^p if for vectors v, w in some dense subspace of \mathcal{H} , the matrix coefficient $\langle \pi(g)v, w \rangle$ is in $L^p(G)$. We also say that the representation is L^{p^+} if the above matrix coefficients are in $L^{p+\varepsilon}(G)$ for every $\varepsilon > 0$. The least p with this property is denoted by $p^+(\pi)$.

The following parameter will be used to control the rate of decay in the mean ergodic theorem and in the asymptotic of the number of lattice points:

$$(3.1) \quad n_\varepsilon(p) = \begin{cases} \text{the least even integer greater than or equal to } p/2, & \text{if } p > 2, \\ 1, & \text{if } p = 2. \end{cases}$$

3.2. A motivating example : lattice points in plane sectors. To illustrate the two ingredients of our approach geometrically, let us consider the example of $G = \mathrm{SL}_2(\mathbb{R})$ acting by isometries on the hyperbolic plane \mathbb{H}^2 (of constant curvature -1) and a lattice Γ in G . Fix a point $o \in \mathbb{H}^2$ and consider a sector with vertex o , namely, the region between two infinite geodesic rays starting at o at an angle $\psi > 0$. We let $S_t(\psi)$ denote the sector intersected with the disc of (hyperbolic) radius t centered at o , and proceed to verify the two conditions stated in the recipe.

1) *Regularity under perturbation, and geometric comparison argument.* It is evident that the uniform probability measure $\sigma_t(\psi)$ supported on $S_t(\psi)$ is dominated by $\frac{2\pi}{\psi}\beta'_t$, where β'_t is the normalised uniform (hyperbolic) measure on a disc of radius t with center o .

Consider now a Cartan polar coordinate decomposition, $G = KA^+K$, where $K = \mathrm{SO}_2(\mathbb{R}) = \{k_\phi : 0 \leq \phi < 2\pi\}$ and $A^+ = \{\mathrm{diag}(e^{s/2}, e^{-s/2}) : s \geq 0\}$. Then the sets $S_t(\psi)$ are given by the coordinates

$$S_t(\psi) = \{k_\phi a_s k_{\phi'} : 0 \leq \phi < 2\pi, 0 \leq s < t, 0 \leq \phi' \leq \psi\}$$

and $S_t(\psi)$ are indeed Lipschitz well-rounded. To see that, first recall the (hyperbolic) cosine formula for triangles in the hyperbolic plane

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \phi.$$

In terms of the Cartan polar coordinates decomposition this formula translates to the fact that $a_t k_\phi a_s = k_1 a_r k_2$ has Cartan component a_r where $\cosh r = \cosh a \cosh b + \sinh a \sinh b \cos \phi$ (see e.g. [N1, §2.2]). This immediately implies Lipschitz control of the radial part a_r in the Cartan decomposition under small perturbations. For the angular part in the Cartan decomposition one need only consider the representation on \mathbb{R}^2 and estimate $(e_1, e_2$ being the standard basis)

$$\begin{aligned} 0 < \langle a_t k_\phi a_s e_1, e_1 \rangle &= e^{t/2} e^{s/2} \langle k_\phi e_1, e_1 \rangle = \langle a_r k_2 e_1, k_1^{-1} e_1 \rangle \\ &\leq \|a_r k_2 e_1\| \leq \|a_r\| = \|a_t k_\phi a_s\| \leq e^{t/2} e^{s/2}. \end{aligned}$$

Hence if $\langle k_\phi e_1, e_1 \rangle \geq 1 - \epsilon$ then the norm of the vector $a_r k_2 e_1$ is at least $(1 - \epsilon) \|a_r\|$ so that $k_2 e_1$ must be close to e_1 . It follows that $\langle k_2 e_1, e_1 \rangle \geq (1 - C\epsilon)$, and similarly for k_1 . Hence the angles of rotation defining the Cartan components k_1 and k_2 are close to zero, and the Cartan components depend in a Lipschitz fashion on the perturbation (see Proposition 7.3 for a general argument).

2) *Spectral estimate.* It is well known that for any lattice Γ in $G = \mathrm{SL}_2(\mathbb{R})$, $\pi_{G/\Gamma}^0$ is an L^{p^+} -representation, where $p^+ = p^+(\Gamma)$. It follows from the spectral transfer principle [N2] (see Section 4 below for a full discussion of the following arguments) that

$$\begin{aligned} \left\| \pi_{G/\Gamma}^0(\sigma_t(\psi)) \right\| &\leq \left(\frac{2\pi}{\psi} \|\lambda_G(\beta_t)\| \right)^{1/n_\epsilon(p)} = \left(\frac{2\pi}{\psi} \Xi_G(t) \right)^{1/n_\epsilon(p)} \\ &= O_\eta \left(\psi^{-1/n_\epsilon(p)} e^{-((2n_\epsilon(p))^{-1} - \eta)t} \right), \quad \eta > 0, \end{aligned}$$

where λ_G denotes the regular representation, and Ξ_G is the Harish-Chandra function (see e.g. [HT, Section 3.1 and Theorem 3.2.1]). The same argument shows that $S_t(\psi)$ satisfy the stable quantitative mean ergodic theorem, so that Theorem 1.9 applies and produces the error term stated there.

More generally, both the stable quantitative mean ergodic theorem and Lipschitz well-roundedness hold for spherical caps in hyperbolic spaces of arbitrary dimension and lead to the following result.

Theorem 3.2. Counting points in sectors in hyperbolic space. *Let \mathbb{H}^m denote hyperbolic m -space (of constant curvature -1) and $S_t(\psi)$ a spherical cap with cone angle ψ (namely intercepting a fraction given by ψ of the area of the unit sphere). Let Γ be any lattice subgroup in $G = \mathrm{SO}^0(m, 1)$ such that $\pi_{G/\Gamma}^0$ is an L^{p^+} -representation. Then the number of lattice points in the spherical cap obeys*

$$\begin{aligned} |\{\gamma \in \Gamma : \gamma o \in S_t(\psi)\}| &= \frac{v_m}{\mathrm{vol}(\Gamma \backslash \mathbb{H}^m)} \psi e^{(m-1)t} \\ &\quad + O_\eta \left(\psi^{-1/n_\epsilon(p)} e^{(m-1) \left(1 - \frac{n_\epsilon(p)-1}{m(m+1)+2} + \eta \right) t} \right), \quad \eta > 0, \end{aligned}$$

with $v_m > 0$ depending only on the dimension m , and the implied constant depending only on $m, p, \varepsilon_0(e, \Gamma)$. (We assume here that only the identity in Γ stabilises o , otherwise the main term should be divided by the size of the stabiliser).

We note that the sectors $S_t(\psi)$ constitute a Lipschitz well-rounded family but it is obviously not an admissible family according to the Definition 1.2, so that the counting results in [GN] do not directly apply.

We refer to [Boc, Mar3, Ni, Sh] for other results on the angular distribution of lattice points in hyperbolic spaces. In particular, in the special case when $m = 2$ and Γ is a principal congruence subgroup of $SL_2(\mathbb{Z})$, an error term in this counting problem was derived by Boca in [Boc], who has also raised the question of which lattices in $SL_2(\mathbb{R})$ satisfy a similar property. Note that Theorem 3.2 applies to general lattice subgroups in hyperbolic spaces, and for tempered lattices $\Gamma \subset SL_2(\mathbb{Z})$ (i.e., when $\pi_{G/\Gamma}^0$ is L^{2+}), the error estimate coincides with the one obtained in [Boc].

4. LATTICE POINTS ON SEMISIMPLE S -ALGEBRAIC GROUPS

4.1. Notation. Let us now set the following notation regarding local fields, algebraic groups and adeles, which will be used throughout the rest of the paper.

Given an algebraic number field F , we denote by V the set of equivalence classes of valuations of F . The set V is the disjoint union $V = V_f \amalg V_\infty$ of the set V_f consisting of non-Archimedean valuations and the set V_∞ consisting of Archimedean valuations. More generally, for $S \subset V$, we also have the decomposition $S = S_f \amalg S_\infty$. For any place $v \in V$, let F_v denote the completion of F w.r.t. the valuation v . Let \mathcal{O} denote the ring of integers in F , and for finite v , let \mathcal{O}_v be its completion, namely $\mathcal{O}_v = \{x \in F_v : v(x) \geq 0\}$. We denote by \mathfrak{p}_v the maximal ideal in \mathcal{O}_v , by $f_v = \mathcal{O}_v/\mathfrak{p}_v$ the residue field, and set $q_v = |f_v|$. As usual, the valuation is normalized by $|s_v|_v = \frac{1}{q_v}$, s_v a uniformizer of \mathcal{O}_v .

We introduce local heights H_v . For an Archimedean local field F_v , and for $x = (x_1, \dots, x_d) \in F_v^d$ we set

$$(4.1) \quad H_v(x) = (|x_1|_v^2 + \dots + |x_d|_v^2)^{1/2},$$

and for a non-Archimedean local field F_v ,

$$(4.2) \quad H_v(x) = \max\{|x_1|_v, \dots, |x_d|_v\}.$$

Let F_v , $v \in S$, be a finite family of (nondiscrete) local fields, and G_v , $v \in S$, be the F_v -points of a semisimple algebraic group \mathbf{G}_v defined over F_v . Let Γ be a lattice in the group $G = \prod_{v \in S} G_v$. We fix representations $\rho_v : G_v \rightarrow \mathrm{GL}_{m_v}(F_v)$, $v \in S$, with finite kernels and index the lattice points according to the height H defined by

$$(4.3) \quad H(g) = \prod_{v \in S} H_v(\rho_v(g_v)), \quad g = (g_v) \in G,$$

where H_v 's are the local heights defined above. We set

$$B_T = \{g \in G : H(g) \leq T\}.$$

4.2. Counting S -integral points. We can now state our solution to the problem of counting S -arithmetic lattice points in S -algebraic groups (and more general lattices in the product).

Theorem 4.1. Counting lattice points in S -algebraic groups: height balls. *Keeping the notation in the previous subsection, assume that the groups \mathbf{G}_v are simply connected, and at least one of \mathbf{G}_v 's is isotropic over F_v (or equivalently, G is noncompact). Fix $\varepsilon_0 > 0$. Then there exists $T_0 > 0$ such that for every*

lattice Γ in G for which the representation $L_0^2(G/\Gamma)$ is L^{p^+} , $x, y \in G$ such that $\varepsilon_0(x, \Gamma), \varepsilon_0(y, \Gamma) \geq \varepsilon_0$, and $T \geq T_0$,

$$|x\Gamma y^{-1} \cap B_T| = \frac{\text{vol}(B_T)}{\text{vol}(G/\Gamma)} + O_\eta \left(\varepsilon_0^{-d^2/(a+d)} \text{vol}(B_T)^{1-(2n_e(p))^{-1}a/(a+d)+\eta} \right)$$

for every $\eta > 0$, where $d = \sum_{v \in S_\infty} \dim(G_v)$ and a is the Hölder exponent of the family $\{B_{e^t}\}$.

Remark 4.2. Let us note the following regarding Theorem 4.1.

- (1) The sets B_{e^t} are always Hölder well-rounded for some $a > 0$ and, in fact, Hölder admissible. As is explained in the proof of Theorem 4.1, this follows from [GN, Theorem 7.19] (see also [BO]). Moreover, if either $S_f = \emptyset$, or $S_\infty = \emptyset$, or for $v \in S_\infty$, the representation ρ_v is self-adjoint (i.e., ${}^t\rho_v(G_v) = \rho_v(G_v)$), we can take the Hölder exponent $a = 1$ (see [GN, Theorem 3.15]).
- (2) The assumption that the representation $\pi_{G/\Gamma}^0$ in $L_0^2(G/\Gamma)$ is L^{p^+} for some $p > 0$ holds, in the set-up of S -algebraic groups, in most cases. See Remark 4.6 below for further discussion.
- (3) If G_v 's are not simply connected, we consider the simply connected covers $\pi : \tilde{G} \rightarrow G$. It is known that $\pi(\tilde{G})$ is normal, and $G/\pi(\tilde{G})$ is Abelian of finite exponent (see [BT]). If Γ is finitely generated, $\Gamma \cap \pi(\tilde{G})$ has finite index in Γ . Applying Theorem 4.1 to the lattice $\tilde{\Gamma} = \pi^{-1}(\Gamma \cap \pi(\tilde{G}))$ in \tilde{G} , one can deduce the asymptotics and the error term for Γ .
- (4) If the height H is bi-invariant under a maximal compact subgroups K_v of G_v when $v \in V_\infty$, we can improve the error estimate by taking $d = \sum_{v \in S_\infty} \dim(K_v \backslash G_v)$ (see also Remark 1.10(2)).
- (5) A further improvement in the error term in Theorem 4.1 can be obtained if in addition the local heights are each bi-invariant under a special maximal compact subgroup (so that the Cartan decomposition holds for G). In this case, we can replace $2n_e(p)$ in the error estimate by p , provided the L^{p^+} -spectrum is uniformly bounded, in the sense defined in [GN, §8.1]. This is indeed often the case.

Taking parts (4) and (5) of the last remark into account, we note that Theorem 4.1 matches the best error estimate established for bi- K -invariant sets in simple higher rank Lie group, but holds in much greater generality. To elucidate this point, we recall the following :

Example 4.3. The most basic example of the non-Euclidean lattice point counting problem is that of integral unimodular matrices, and balls w.r.t. the Hilbert-Schmidt norm, which is the Archimedean valuation. In this case, taking parts (4) and (5) of the foregoing remark into account, we have $d = \dim \text{SL}_m(\mathbb{R})/\text{SO}_m(\mathbb{R}) = m(m+1)/2 - 1$, the balls are Lipschitz so $a = 1$, and the integrability parameter is $p^+ = 2(m-1)$ namely the representation in $L_0^2(\text{SL}_m(\mathbb{R})/\text{SL}_m(\mathbb{Z}))$ is $L^{2(m-1)^+}$ (see [DRS]). Theorem 4.1 then implies :

$$|\text{SL}_m(\mathbb{Z}) \cap B_T| = \frac{\text{vol}(B_T)}{\text{vol}(\text{SL}_m(\mathbb{R})/\text{SL}_m(\mathbb{Z}))} + O_\eta \left(\text{vol}(B_T)^{1-1/(m^3-m)+\eta} \right), \quad \eta > 0.$$

The latter estimate coincides with the best current error term obtained in [DRS] for this case.

Another natural family of balls on an S -algebraic group $G = \prod_{v \in S} G_v$ is defined with respect to standard $CAT(0)$ metrics on the corresponding symmetric spaces and buildings. Let X_v denote the symmetric space of G_v if it is Archimedean and the Bruhat–Tits building of G_v otherwise. For fixed $x = (x_v) \in \prod_{v \in S} X_v$, we set

$$d(g) = \left(\sum_{v \in S} d_v(gx_v, x_v)^2 \right)^{1/2}$$

where d_v are the standard metrics on X_v . Let

$$(4.4) \quad B_t = \{g \in G : d(g) \leq t\}.$$

Our method allows to deal with lattice subgroups of G which are not necessarily irreducible. This is related to fact that the sets B_t are well-balanced (see [GN, Definition 3.17]). Namely, the volume of the sets B_t does not concentrate along proper direct factors of G (see [GN, Theorem 3.18]). The error term in the lattice counting problem can be estimated in terms the relative volume growth

$$r = \max_{L < G} \limsup_{t \rightarrow \infty} \frac{\log m_L(B_t \cap L)}{\log m_G(B_t)}$$

where the maximum is taken over proper direct factors L of G . If all the factors G_v are not compact, then $r < 1$ by [GN, Theorem 3.18].

Theorem 4.4. Counting lattice points in S -algebraic groups: metric balls.

Let G be as in Theorem 4.1, with all factors G_v non-compact. For every $\varepsilon_0 > 0$ there exists $t_0 > 0$ such that for every lattice Γ in G for which the representation of G_v , $v \in S$, on the orthogonal complement of $L^2(G/\Gamma)^{G_v}$ is L^{p^+} , $x, y \in G$ such that $\varepsilon_0(x, \Gamma), \varepsilon_0(y, \Gamma) \geq \varepsilon_0$, and $t \geq t_0$,

$$|x\Gamma y^{-1} \cap B_t| = \frac{\text{vol}(B_t)}{\text{vol}(G/\Gamma)} + O_\eta \left(\varepsilon_0^{-d^2/(1+d)} \text{vol}(B_t)^{1-(1-\sqrt{3r^2+1}/2)/n_\varepsilon(p)(1+d)+\eta} \right)$$

for every $\eta > 0$, where $d = \sum_{v \in S_\infty} \dim(X_v)$.

According to our recipe, to prove Theorems 4.1 and 4.4 we need to establish a decay estimate for the operator norms of the averages β_t supported on the sets B_t , and establish the Lipschitz well-roundedness of the balls.

Turning to the first ingredient, we now show that the stable quantitative mean ergodic theorem for G holds in great generality.

Theorem 4.5 ([N2], see also [GN]). **Stable mean ergodic theorem for S -algebraic groups.**

Assume that the groups G_v are simply connected, and at least one of G_v 's is isotropic over F_v . Consider an action of G on a standard Borel probability space (X, μ) and assume that the corresponding representation π_X^0 of G on the orthogonal complement of $L^2(X)^G$ is L^{p^+} . Let β be an absolutely continuous probability measure on G such that $\|\beta\|_q < \infty$ for some $q \in [1, 2)$. Then

$$\|\pi_X^0(\beta)\| \leq C_q \|\beta\|_q^{1/n_\varepsilon(p)}$$

where $n_\varepsilon(p)$ is defined in (3.1).

When β_t are the uniform averages supported on the sets B_t , we have

$$\|\pi_X^0(\beta_t)\| \leq C'_\eta m_G(B_t)^{-(2n_\varepsilon(p))^{-1}+\eta}, \quad \eta > 0.$$

In particular, if the family B_t is Hölder well-rounded and the action of G on (X, μ) is ergodic, the stable quantitative mean ergodic theorem holds with the above rate.

Proof. We recall the spectral transfer principle from [N2]. By Jensen's inequality, for real-valued functions $f_1, f_2 \in (L^2(X)^G)^\perp$,

$$\begin{aligned} \langle \pi_X^0(\beta)f_1, f_2 \rangle^{n_e} &= \left(\int_G \langle \pi_X^0(g)f_1, f_2 \rangle d\beta(g) \right)^{n_e} \leq \int_G \langle \pi_X^0(g)f_1, f_2 \rangle^{n_e} d\beta(g) \\ &= \int_G \langle (\pi_X^0)^{\otimes n_e}(g)f_1^{\otimes n_e}, f_2^{\otimes n_e} \rangle d\beta(g) = \langle (\pi_X^0)^{\otimes n_e}(\beta)f_1^{\otimes n_e}, f_2^{\otimes n_e} \rangle \\ &\leq \|(\pi_X^0)^{\otimes n_e}(\beta)\| \|f_1\|^{n_e} \|f_2\|^{n_e}. \end{aligned}$$

This implies that

$$\|\pi_X^0(\beta)\| \leq \|(\pi_X^0)^{\otimes n_e}(\beta)\|^{1/n_e}.$$

It is easy to see that $(\pi_X^0)^{\otimes n_e}$ is an L^{2^+} -representation. Hence, it follows from [CHH] that $(\pi_X^0)^{\otimes n_e}$ is weakly contained in the regular representation λ_G of G , and

$$\|(\pi_X^0)^{\otimes n_e}(\beta)\| \leq \|\lambda_G(\beta)\|.$$

We can now estimate $\|\lambda_G(\beta)\|$ using the Kunze-Stein inequality, namely

$$\|\beta * f\|_2 \leq C_q'' \|\beta\|_q \|f\|_2, \quad q \in [1, 2),$$

for every $f \in L^2(G)$. This inequality was proved for Archimedean semisimple groups by Cowling [Co2] and for non-Archimedean semisimple simply connected groups by Veca [V]. Clearly, a product of Kunze-Stein groups is a Kunze-Stein group. Indeed, if β is a product function, the estimate for its norm as a convolution operator follows immediately by considering product functions f . Any β is the $L^q(G)$ -norm limit of a sequence of product functions, and the estimate follows. Hence every semisimple simply connected S -algebraic group satisfies the Kunze-Stein inequality. This implies that

$$\|\lambda_G(\beta)\| \leq C_\eta'' m_G(B)^{-1/2+\eta}, \quad \eta > 0,$$

and the desired norm estimate follows.

The second claim in Theorem 4.5, namely the stable mean ergodic theorem for families B_t is an immediate consequence of the previous inequality, since the implied constant is uniform for all sets $B_t^\pm(\varepsilon)$. \square

Remark 4.6. Assume that the component groups G_v are simply connected. Then Theorem 4.5 applies, in particular, to the actions listed below.

- (i) G is a Kazhdan group, and (X, μ) is any ergodic G -space.
- (ii) $X = G/\Gamma$ where G is an almost simple connected Lie group, and Γ is any lattice.
- (iii) $X = G/\Gamma$ where Γ is an irreducible congruence subgroup in an S -arithmetic lattice of a semisimple S -algebraic group (see Section 5 for notation). Moreover, the parameter $p^+ = p^+(\Gamma)$ is then bounded above uniformly over all congruence subgroups (namely property τ holds).
- (iv) $X = G/\Gamma$ where G is a connected semisimple Lie group all of whose factors are locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$, and Γ is any irreducible lattice.
- (v) $X = G/\Gamma$ with G as in (iii) and Γ any lattice commensurable with an irreducible congruence lattice.

Verification of these claims depends on the fact that matrix coefficients of nontrivial irreducible representations π of almost simple simply connected groups are in L^p for some $p = p(\pi)$ (see [BW, Co1, Ho, HM, Li, LZ, Oh]). This implies that $L_0^2(X)$

is an L^p -representation for some $p > 0$ provided that it has strong spectral gap, i.e., no noncompact simple factor of G has almost invariant vectors. Hence, it remains to check that in (ii)–(v), one has the strong spectral gap. Now (ii) follows from the work of Borel and Garland [BG], (iii) follows from the work of Clozel [Cl], (iv) was recently proved by Kelmer and Sarnak [KS], and (v) follows from (iii) and [KM, Lemma 3.1].

We now complete the proof of Theorems 4.1 and 4.4 following the recipe of §3.1.

Proof of Theorem 4.1. For $v \in S_f$, the local heights are bi-invariant under a compact open subgroups \mathcal{O}^v of G_v . For $v \in S_\infty$, we set

$$\mathcal{O}_\varepsilon^v = \{g \in G_v : H_v(\rho_v(g_v^{\pm 1}) - id) < \varepsilon\}.$$

Then the family of symmetric neighbourhoods

$$\mathcal{O}_\varepsilon = \prod_{v \in S_\infty} \mathcal{O}_\varepsilon^v \times \prod_{v \in S_f} \mathcal{O}^v$$

has local dimension d , which equals the real dimension of G_∞ . Using that for every $x_1, x_2 \in M_{n_v}(K_v)$,

$$H_v(x_1 x_2) \leq H_v(x_1) H_v(x_2),$$

we deduce that for $g, h \in \mathcal{O}_\varepsilon$ and $b \in B_T$,

$$H(gbh) \leq H(g)H(b)H(h) \leq (1 + \varepsilon)^{2|S_\infty|} H(b) \leq (1 + \varepsilon)^{2|S_\infty|} T.$$

Hence,

$$(4.5) \quad B_T^+(\varepsilon) = \mathcal{O}_\varepsilon B_T \mathcal{O}_\varepsilon \subset B_{(1+\varepsilon)^{2|S_\infty|} T}.$$

Similarly,

$$(4.6) \quad B_T^-(\varepsilon) \subset B_{(1+\varepsilon)^{-2|S_\infty|} T}.$$

Now, if H_v are constant on G_v for $v \in S_\infty$, then it is clear that the family $\{B_{e^t}\}$ is even Lipschitz admissible. Otherwise, the function $t \mapsto \log \text{vol}(B_{e^t})$ is uniformly Hölder (see [GN, Theorem 7.19] or [BO]), and it follows from (4.5) and (4.6) that the family $\{B_{e^t}\}$ is Hölder admissible (and, in particular, Hölder well-rounded). Hence, combining Corollary 1.11 and Theorem 4.5, the result follows. \square

Proof of Theorem 4.4. Let $\mathcal{O}_\varepsilon = \{g \in G : d(g) < \varepsilon\}$. It follows from the triangle inequality that

$$B_t^+(\varepsilon) \subset B_{t+\varepsilon} \quad \text{and} \quad B_t^-(\varepsilon) \supset B_{t-\varepsilon}.$$

If $S_\infty \neq \emptyset$, then the function $t \mapsto \log m_G(B_t)$ is uniformly Lipschitz by [GN, Theorem 3.18]. Hence, the family B_t is Lipschitz well-rounded in this case, and in fact Lipschitz admissible. Otherwise, the family B_t is bi-invariant under a compact open subgroup of G and, in particular, Lipschitz well-rounded as well.

In view of Corollary 1.11 (and Remark 1.10(2)), it remains to prove the quantitative mean ergodic theorem for the uniform averages β_t along the sets B_t . Namely, we need to show that for every $f \in L_0^2(X)$,

$$(4.7) \quad \|\pi_{G/\Gamma}^0(\beta_t)f\|_2 \ll_\eta m_G(B_t)^{(1-\sqrt{3r^2+1}/2)/n_\varepsilon(p)+\eta} \|f\|_2, \quad \eta > 0.$$

For $J \subset S$, we set $G_J = \prod_{j \in J} G_j$ and $G^J = \prod_{j \notin J} G_j$. We observe that $L^2(X) = \sum_{J \subset I} \mathcal{H}_J$ where \mathcal{H}_J are orthogonal closed G -invariant subspaces of $L^2(X)$ such that every vector in \mathcal{H}_J is fixed by G_J and there are no nonzero vectors fixed by G_j , $j \notin J$. We note that the representation of G^J on \mathcal{H}_J is L^{p^+} . Indeed, the

representation of each G_j , $j \notin J$, on \mathcal{H}_J is L^{p^+} , and every irreducible representation of G^J appearing in the decomposition of \mathcal{H}_J is a tensor product of irreducible representations of the factors.

For $f \in \mathcal{H}_J$ and $J \neq S$,

$$\pi_{G/\Gamma}^0(\beta_t)f(x) = \frac{1}{m_G(B_t)} \int_{B_t^J} m_{G^J}(B_{J,\sqrt{t^2-d(g)^2}})f(g^{-1}x) dm_{G^J}(g) = (\beta_t * f)(x)$$

where $\beta_t(g) = \frac{1}{m_G(B_t)} m_{G^J}(B_{J,\sqrt{t^2-d(g)^2}}) \chi_{B_t^J}$. By Theorem 4.5, for every $q \in [1, 2)$,

$$(4.8) \quad \|\pi_{G/\Gamma}^0(\beta_t)f\|_2 \ll_q \|\beta_t\|_q^{1/n_\epsilon(p)} \|f\|_2.$$

We have

$$(4.9) \quad \|\beta_t\|_q = \frac{1}{m_G(B_t)} \left(\int_{B_t^J} m_{G^J}(B_{J,\sqrt{t^2-d(g)^2}})^q dm_{G^J}(g) \right)^{1/q}.$$

Let

$$v_J = \lim_{t \rightarrow \infty} \frac{1}{t} \log m_{G^J}(B_{J,t}) \quad \text{and} \quad v^J = \lim_{t \rightarrow \infty} \frac{1}{t} \log m_{G^J}(B_t^J).$$

This limits exist by [GN, Lemma 7.11]. We have

$$(4.10) \quad m_G(B_{J,t}) \ll_\eta e^{(v_J+\eta)t} \quad \text{and} \quad m_G(B_t^J) \ll_\eta e^{(v^J+\eta)t}$$

for all $\eta > 0$ and $t \geq 0$, and

$$(4.11) \quad m_G(B_{J,t}) \gg_\eta e^{(v_J-\eta)t} \quad \text{and} \quad m_G(B_t^J) \gg_\eta e^{(v^J-\eta)t}$$

for all $\eta > 0$ and $t \geq t(\eta)$.

We claim that for $\eta > 0$ and $t \geq 0$,

$$(4.12) \quad \int_{B_t^J} m_{G^J}(B_{J,\sqrt{t^2-d(g)^2}})^q dm_{G^J}(g) \ll_\eta \exp\left(\left(\sqrt{(qv_J)^2 + (v^J)^2} + \eta\right)t\right),$$

and for $\eta > 0$ and $t \geq t(\eta)$,

$$(4.13) \quad \int_{B_t^J} m_{G^J}(B_{J,\sqrt{t^2-d(g)^2}})^q dm_{G^J}(g) \gg_\eta \exp\left(\left(\sqrt{(qv_J)^2 + (v^J)^2} - \eta\right)t\right).$$

To verify these claims, we need to consider two cases: when G^J has at least one Archimedean factor, and when G^J consists of non-Archimedean factors.

In the first case, we observe that the sets B_t^J are admissible by [GN, Theorem 3.18] and by [GN, Proposition 3.13], we have $m_{G^J} = \int_0^\infty m_t^J dt$ where m_t^J be a measure supported on $S_t^J = \{g \in G^J : d(g) = t\}$. Moreover, it follows from the admissibility that

$$(4.14) \quad m_t^J(S_t^J) \ll_\eta e^{(v^J+\eta)t}$$

for all $\eta > 0$ and $t \geq 0$. Then by (4.10) and (4.14),

$$\begin{aligned} \int_{B_t^J} m_{G^J}(B_{J,\sqrt{t^2-d(g)^2}})^q dm_{G^J}(g) &= \int_0^1 t m_{G^J}(B_{J,t\sqrt{1-u^2}})^q m_{tu}^J(S_{tu}^J) du \\ &\ll_\eta \int_0^1 t \exp((qv_J\sqrt{1-u^2} + v^J u + \eta)t) du \\ &\ll_\eta \exp\left(\left(\sqrt{(qv_J)^2 + (v^J)^2} + \eta\right)t\right) \end{aligned}$$

for every $\eta > 0$, where the last estimate is obtained by maximising the function $\phi(u) = qv_J\sqrt{1-u^2} + v^J u$. This proves (4.12). To prove the opposite inequality, we note that

$$m_t^J(S_t^J) \gg_\eta e^{(v^J - \eta)t}$$

for all $\eta > 0$ and $t \geq t(\eta)$ (see [GN, Proof of Theorem 3.18]). Taking a sufficiently small neighbourhood U of the point $u_0 \in (0, 1)$ of maximum of the function ϕ , we obtain

$$\begin{aligned} \int_{B_t^J} m_{G^J}(B_{J, \sqrt{t^2 - d(g)^2}})^q dm_{G^J}(g) &\geq \int_U t m_{G^J}(B_{J, t\sqrt{1-u^2}})^q m_{tu}^J(S_{tu}^J) du \\ &\gg_\eta \exp\left(\left(\sqrt{(qv_J)^2 + (v^J)^2} - \eta\right)t\right) \end{aligned}$$

for every $\eta > 0$, which proves (4.13).

In the case when G^J is a product of non-Archimedean factors, we have

$$\int_{B_t^J} m_{G^J}(B_{J, \sqrt{t^2 - d(g)^2}})^q dm_{G^J}(g) = \sum_{u \in [0, 1]: m_{G^J}(S_{tu}^J) \neq \emptyset} m_{G^J}(B_{J, t\sqrt{1-u^2}})^q m_{G^J}(S_{tu}^J).$$

Since $|d(G^J) \cap [0, t]| \ll t^d$ for some $d > 0$ and $t \geq 1$, the inequality (4.12) follows from (4.10). Since the gaps between distances $d(G^J)$ are uniformly bounded, there exists $u_t \in (0, 1)$ such that $S_{tu_t}^J \neq \emptyset$ and $|u_t - u_0| = O(1/t)$. Since $u_0 \in (0, 1)$, we have $|\phi(u_t) - \phi(u_0)| = O(1/t)$. As in [GN, Lemma 7.11], when $S_t^J \neq \emptyset$, we have

$$m^J(S_t^J) \gg_\eta e^{(v^J - \eta)t}$$

for all $\eta > 0$ and $t \geq t(\eta)$. Then

$$\begin{aligned} \int_{B_t^J} m_{G^J}(B_{J, \sqrt{t^2 - d(g)^2}})^q dm_{G^J}(g) &\gg_\eta \exp\left(\left(qv_J\sqrt{1-u_t^2} + v^J u_t - \eta\right)t\right) \\ &\geq \exp\left(\left(qv_J\sqrt{1-u_0^2} + v^J u_0 - O(1/t) - \eta\right)t\right) \end{aligned}$$

for every $\eta > 0$. This implies (4.13).

Since

$$m_G(B_t) = \int_{B_t^J} m_{G^J}(B_{J, \sqrt{t^2 - d(g)^2}}) dm_{G^J}(g),$$

the estimates (4.12) and (4.13) imply that

$$v := \lim_{t \rightarrow \infty} \frac{1}{t} \log m_G(B_t) = \sqrt{(v_J)^2 + (v^J)^2}.$$

Setting $r_J = v_J/v$, we obtain

$$\begin{aligned} \left(\int_{B_t^J} m_{G^J}(B_{J, \sqrt{t^2 - d(g)^2}})^q dm_{G^J}(g)\right)^{1/q} &\ll_\eta \exp\left(\left(\sqrt{(v_J)^2 + (v^J/q)^2} + \eta\right)t\right) \\ &\ll_\eta m_G(B_t) \sqrt{r_J^2 + q^{-2}(1-r_J^2) + \eta} \end{aligned}$$

for every $\eta > 0$.

Finally, it follows from (4.8) and (4.12) that

$$\|\pi_{G/\Gamma}^0(\beta_t) f\|_2 \ll_\eta m_G(B_t)^{\left(1 - \sqrt{r_J^2 + (1-r_J^2)/q^2}\right)/n_e(p) + \eta} \|f\|_2, \quad \eta > 0.$$

Since this estimate holds for every $q \in [1, 2)$, the claim (4.7) follows. This completes the proof of the theorem. \square

5. CONGRUENCE SUBGROUPS AND DENSITY HYPOTHESIS

Let $\mathbf{G} \subset \mathrm{GL}_m$ be a connected semisimple algebraic group defined over a number field F . We fix a finite set S of places of F which contains all Archimedean places V_∞ and the group $G = \prod_{v \in S} \mathbf{G}(F_v)$ is noncompact. Then $\Gamma = \mathbf{G}(\mathcal{O}_S)$, where \mathcal{O}_S is the ring of S -integers, is a lattice in G . Given an ideal \mathfrak{a} of \mathcal{O}_S , we introduce a congruence subgroup

$$\Gamma(\mathfrak{a}) = \{\gamma \in \Gamma : \gamma = I \pmod{\mathfrak{a}}\}.$$

The height function H on G is defined as in (4.3) and $B_T = \{g \in G : H(g) < T\}$. If \mathbf{G} is simply connected and F -simple, then property (τ) , established in full generality by Clozel [Cl], shows that there exists $p > 0$ such that all representations in $L_0^2(G/\Gamma(\mathfrak{a}))$ as \mathfrak{a} varies are L^{p^+} -representations, with p^+ independent of \mathfrak{a} . Hence, Theorem 4.1 immediately implies the following uniformity result in counting lattice points, generalising [NS].

Theorem 5.1. Uniformity in counting over congruence groups. *Keeping the notation of the previous paragraph, if \mathbf{G} is simply connected and F -simple, there exists $T_0 > 0$ such that for every $\gamma_0 \in \Gamma$, all ideals \mathfrak{a} of \mathcal{O}_S , and $T \geq T_0$,*

$$|\{\gamma \in \gamma_0\Gamma(\mathfrak{a}) : H(\gamma) < T\}| = \frac{\mathrm{vol}(B_T)}{[\Gamma : \Gamma(\mathfrak{a})]} + O_\eta \left(\mathrm{vol}(B_T)^{1-(2n_\epsilon(p))^{-1}a/(a+d)+\eta} \right)$$

for every $\eta > 0$. Here the measure on G is normalised so that $\mathrm{vol}(G/\Gamma) = 1$, a is the Hölder exponent for the family $\{B_{e^t}\}$, $d = \sum_{v \in V_\infty} \dim \mathbf{G}(F_v)$, and the implied constant is independent of the ideal \mathfrak{a} .

Let us now recall the following conjecture:

Conjecture 5.2 ([SX],[Sa]). *For any semisimple algebraic \mathbb{Q} -group $\mathbf{G} \subset \mathrm{GL}_m$, the following upper bound holds, uniformly in $N \in \mathbb{N}$ (for any fixed choice of norm)*

$$|\{\gamma \in \Gamma(N) : \|\gamma\| < T\}| = O_\eta \left(\frac{T^{\alpha+\eta}}{[\Gamma(1) : \Gamma(d)]} + T^{\alpha/2} \right), \quad \eta > 0,$$

where $\Gamma(N)$ are the principal congruence group mod N in $\mathbf{G}(\mathbb{Z})$, and

$$\alpha = \limsup_{T \rightarrow \infty} \frac{\log \mathrm{vol}(B_T)}{\log T}.$$

From Theorem 5.1, we obtain the following result :

Corollary 5.3. *For any semisimple \mathbb{Q} -simple algebraic group $\mathbf{G} \subset \mathrm{GL}_m$, and any fixed choice of norm,*

$$|\{\gamma \in \Gamma(N) : \|\gamma\| < T\}| = O_\eta \left(\frac{T^{\alpha+\eta}}{[\Gamma(1) : \Gamma(N)]} + T^{(\alpha+\eta)(1-\theta)} \right), \quad \eta > 0,$$

where $\theta = (2n_\epsilon(p))^{-1} / (1 + \dim(\mathbf{G}(\mathbb{R})/K))$ and K is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. Here $N \in \mathbb{N}$ is arbitrary, and the implied constant is independent of N . α is the rate of volume growth of the norm balls.

Proof. To deduce this corollary from Theorem 5.1, we consider the simply connected cover $\pi : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$ and note that $\pi(\tilde{\mathbf{G}}(\mathbb{Z}))$ is commensurable to $\mathbf{G}(\mathbb{Z})$. Hence, without loss of generality, we may assume that \mathbf{G} is simply connected.

With respect to a suitable basis of \mathbb{R}^d , \mathbf{G} is self-adjoint and there exists a maximal compact subgroup $K \subset \mathbf{G}(\mathbb{R})$ such that $K \subset \mathrm{SO}_m(\mathbb{R})$. We note that the estimate in the theorem is independent of a choice of the norm. Hence, we may assume that $\|\cdot\|$ is a Euclidean norm with respect to the above basis. Then the sets $\{g \in \mathbf{G}(\mathbb{R}) : \|g\| \leq e^t\}$ are Lipschitz admissible (see [GN, Theorem 3.15]), and bi- K -invariant. Hence, the corollary follows from Theorem 4.1 and Remarks 4.2(1),(4). \square

Comparing Corollary 5.3 and Conjecture 5.2, let us note the following.

- (1) The second term appearing in the estimate stated in Conjecture 5.2 is the best possible, and is asserted only for the *principal* congruence groups. It may fail if more general finite-index subgroups are admitted as demonstrated by the construction of exceptional eigenvalues in [BS], [BLS]. Thus the conjecture predicts a regularity property of the lattice point counting problem satisfied specifically by principal congruence subgroups.
- (2) The proof of Theorem 1.9 above can not produce the second term called for in Conjecture 5.2, which is the square root of the volume of the ball. Indeed, Theorem 1.9 will still yield an error term greater than the square root of the volume even if the spectral gap is the largest possible, namely all the representations occurring in $L_0^2(\mathbf{G}(\mathbb{R})/\Gamma(N))$ are tempered. An error term with this quality can be established only for a smooth weighted form of the lattice point counting problem, see [NS]. On the upside, Theorem 5.1 actually gives an error estimate, uniform over all $\Gamma(N)$ and their cosets, namely a lower bound as well as an upper bound.

The cases where Conjecture 5.2 has been verified are the set of arithmetic lattices in $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ [SX, Thm. 1]. Note that in those cases the conjecture was established *without* assuming a spectral gap, and indeed was used to derive it, thus giving an independent approach to uniform spectral gaps for congruence subgroups [SX, Corollary 2].

An important application of Conjecture 5.2 is to the density hypothesis, which bounds the multiplicities of the $\mathbf{G}(\mathbb{R})$ -representations occurring in $L^2(\mathbf{G}(\mathbb{R})/\Gamma(N))$ (see [DG-W1][DG-W2] for a discussion of this problem). For an irreducible non-trivial π , we let $p_K^\pm(\pi)$ denote the infimum over $p \geq 2$ such that the K -finite matrix coefficients of π are in $L^p(\mathbf{G}(\mathbb{R}))$. Let $m(\pi, \Gamma(N))$ denote the multiplicity in which π occurs in $L^2(\mathbf{G}(\mathbb{R})/\Gamma(N))$. Consider the following “density hypothesis”:

Conjecture 5.4 ([SX], [Sa]). *With notation as in Conjecture 5.2, assume that $\mathbf{G}(\mathbb{Z})$ is cocompact. Then for all $\eta > 0$,*

$$m(\pi, \Gamma(N)) = O_\eta \left([\Gamma(1) : \Gamma(N)]^{(2/p_K^\pm(\pi)) + \eta} \right).$$

When G has real rank one and Γ is cocompact, the density hypothesis was shown to follow from Conjecture 5.2 (see [SX]).

Remark 5.5. The method used in [SX] can be combined with Theorem 5.1 to give an alternative proof of a result in [SX], which states that the estimate in Conjecture 5.4 holds with the power $2/p_K(\pi)$ of $[\Gamma(1) : \Gamma(N)]$ replaced by a weaker estimate. However, since Theorem 5.1 does not require the rank-one hypothesis, one expects

that it can be used to establish a multiplicity bound in terms of an appropriate power of $[\Gamma(1) : \Gamma(N)]$ more generally, for groups of arbitrary rank.

6. RATIONAL POINTS, AND KUNZE-STEIN PHENOMENON ON ADELE GROUPS

Let F be an algebraic number field. Keeping the notation from §4.1, we define the height of an F -rational vector $x = (x_1, \dots, x_d) \in F^d$ by

$$H(x) = \prod_{v \in V} H_v(x),$$

where the local heights H_v are defined as in (4.1)–(4.2). For example, if $F = \mathbb{Q}$ and $x \in \mathbb{Q}^\times \cdot (x_1, \dots, x_d)$ where $x_1, \dots, x_d \in \mathbb{Z}$ and $\gcd(x_1, \dots, x_d) = 1$, then

$$H(x) = (|x_1|^2 + \dots + |x_d|^2)^{1/2}.$$

The number of rational points with bounded height lying on a projective variety is finite, and one of the fundamental problems in arithmetic geometry is to determine its asymptotics (see, for instance, [Ts]).

Let $\mathbf{G} \subset \mathrm{GL}_m$ be a semisimple algebraic group defined over F . Then the cardinality of the set $\{\gamma \in \mathbf{G}(F) : H(\gamma) \leq T\}$ is finite, and we are interested in its asymptotic as $T \rightarrow \infty$. The set $\mathbf{G}(F)$ embeds discretely in the group $\mathbf{G}(\mathbb{A})$ of adèles as a subgroup of finite covolume, and the height H extends to $\mathbf{G}(\mathbb{A})$. We set

$$(6.1) \quad B_T = \{g \in \mathbf{G}(\mathbb{A}) : H(g) \leq T\}.$$

To state our main result, we note that it follows from [Cl] that provided \mathbf{G} is simply connected and F -simple, the representation $\pi_{\mathbf{G}(\mathbb{A})/\mathbf{G}(F)}^0$ in $L_0^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F))$ is L^{p^+} for some $p > 0$ (we will explain this in detail in the proof of Theorem 6.1 below).

Theorem 6.1. *Assume that the group \mathbf{G} is simply connected and F -simple. Then*

$$|\mathbf{G}(F) \cap B_T| = \frac{m_{\mathbf{G}(\mathbb{A})}(B_T)}{m_{\mathbf{G}(\mathbb{A})}(\mathbf{G}(\mathbb{A})/\mathbf{G}(F))} + O_\eta \left(m_{\mathbf{G}(\mathbb{A})}(B_T)^{1 - (2n_\epsilon(p))^{-1}a/(2d+2a)+\eta} \right)$$

for every $\eta > 0$, where a is the Hölder exponent of the family $\{B_{e^t}\}$ and $d = \sum_{v \in V_\infty} \dim_{\mathbb{R}} \mathbf{G}(F_v)$.

Remark 6.2. Let us note the following regarding Theorem 6.1 (compare with Remark 4.2).

- (1) If the height $\prod_{v \in V_\infty} H_v$ is bi-invariant under a maximal compact subgroup K , then d in the error estimate can be replaced by $d - \dim_{\mathbb{R}} K$.
- (2) If in addition the local heights H_v are each bi-invariant under a special maximal compact subgroup K_v of G_v , the error term in Theorem 6.1 can be improved by replacing $2n_\epsilon(p)$ by p , provided the L^{p^+} -spectrum is uniformly bounded in the sense of [GN, §8.1].
- (3) If $\mathbf{G} \subset \mathrm{GL}_m$ is self-adjoint (namely invariant under transpose) then the family B_{e^t} are Lipschitz well-rounded. Indeed, this is clear when the height $\prod_{v \in V_\infty} H_v$ is constant on G_∞ . Otherwise, we have Lipschitz estimate at the Archimedean places [GN, Proposition 7.5], and [GN, Theorem 3.15] (or the argument in [GO2, Proposition 2.19(2)]) yield the desired Lipschitz estimate. Then in Theorem 6.1, one can set $a = 1$.

We note that the main term of the asymptotics of the number of rational points on semisimple group varieties was computed in [STBT2] (using direct spectral expansion of the automorphic kernel) and in [GMO] (using mixing). However, as noted in §1.3, these methods do not produce an error term of the same quality as Theorem 6.1.

6.1. The structure of semisimple adèle groups. The adèle group $G = \mathbf{G}(\mathbb{A})$ is defined as the direct product $G = G_\infty \times G_f$, where $G_\infty = \prod_{v \in V_\infty} G_v$, and G_f is the restricted direct product $\prod'_{v \in V_f} (G_v, K_v)$ of the locally compact groups $\mathbf{G}(F_v)$ w.r.t. the compact open subgroups K_v , which for almost all $v \in V_f$ satisfies $K_v = \mathbf{G}(\mathcal{O}_v)$ (for a fixed integral model for \mathbf{G}). Thus each element of the restricted direct product G_f can be identified with a sequence $(g_v)_{v \in V_f}$, such that $g_v \in \mathbf{G}(\mathcal{O}_v)$ for almost every $v \in V_f$, and G is a locally compact σ -compact group. We recall that if we choose Haar measures m_v on each G_v , normalised so that $m_v(K_v) = 1$ for $v \in V_f$, and define the measure m_{G_f} via the construction of restricted product of measure spaces, namely

$$\left(G_f, \prod_{v \in V_f} K_v, m_{G_f} \right) = \prod'_{v \in V_f} (G_v, K_v, m_v),$$

then m_{G_f} is a Haar measure on G_f (see [Bl] and [Mo] for further details on this construction). Haar measure on $G = G_\infty \times G_f$ is then the direct product $m_{G_\infty} \times m_{G_f}$.

Now assume that \mathbf{G} is a semisimple simply connected algebraic group defined over F . We choose the family of subgroups K_v so that an analogue of the Iwasawa decomposition holds for G . Recall that by [Ti], for almost all v , $\mathbf{G}(\mathcal{O}_v)$ is hyperspecial maximal compact subgroup of G_v . For every $v \in V$, we fix a maximal compact subgroup K_v of G_v so that K_v is special for all $v \in V_f$ and $K_v = \mathbf{G}(\mathcal{O}_v)$ for almost all v . Then for every v , the Iwasawa decomposition $G_v = K_v P_v$ holds where P_v is a closed amenable subgroup given by $P_v = \mathbf{Z}(K_v) \mathbf{U}(K_v)$ where \mathbf{Z} is the centraliser of a suitable maximal F_v -split torus in \mathbf{G} , and \mathbf{U} is the subgroup generated by positive root groups (see [Ti] in the non-Archimedean case). Setting $K = \prod_{v \in V} K_v$ and $P = \prod'_{v \in V} (P_v, P_v \cap K_v)$, we have the Iwasawa decomposition $G = KP$ for the adèle group. For $g \in G$, we denote by $p(g)$ the P -component of g with respect to the Iwasawa decomposition. The element $p(g)$ is well-defined modulo $P \cap K$, and the modular function of P is constant on each coset of $P \cap K$.

6.2. Harish-Chandra function. The Harish-Chandra function is the K -bi-invariant function on G defined by

$$\Xi_G(g) = \int_K \Delta_P(p(gk))^{-1/2} dk$$

where Δ_P is the modular function of P .

Ξ_G plays fundamental role in analysis on semisimple groups over the adèles. First, let us note that $\Xi_G(g) = \prod_{v \in V} \Xi_{G_v}(g_v)$, since $\Delta_P(p) = \prod_{v \in V} \Delta_{P_v}(p_v)$, $K = \prod_{v \in V} K_v$, and Haar probability measure on K is the product of the Haar probability measures on K_v , $v \in V$. Second, note that the Cowling–Haagerup–Howe argument [CHH], which is valid for every group with an Iwasawa decomposition (see [GN, §5.1]), shows that matrix coefficients of K -finite vectors of a unitary representation

π which is weakly contained in the regular representation are estimated by

$$(6.2) \quad \langle \pi(g)\xi, \eta \rangle \leq \sqrt{(\dim \langle K\xi \rangle)(\dim \langle K\eta \rangle)} \|\xi\| \|\eta\| \Xi_G(g)$$

Although the Harish-Chandra function for semisimple groups over local fields is in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$, this is no longer the case for the group of adèle points. Instead, we have the following result.

Proposition 6.3. Integrability of the Harish Chandra function on adèle groups. *Keeping the assumption and notation of the previous subsection, $\Xi_G \in L^{4+\varepsilon}(G)$ for every $\varepsilon > 0$, where $G = \mathbf{G}(\mathbb{A})$.*

Let us immediately note that for general simple groups over the adèles the exponent 4 is the best possible. In particular such groups do not satisfy the standard Kunze-Stein inequality, which requires that the integrability exponent be equal to 2, but only a weaker version of it (see Theorem 6.6 below).

Proposition 6.4. Optimality of the integrability exponent. *The exponent 4 in Proposition 6.3 is optimal for $\mathbf{G} = \mathrm{PGL}_2$.*

Proof. Let $a_v = \mathrm{diag}(s_v, 1)$ where s_v denotes the uniformiser of F_v . We have the Cartan decomposition

$$G_v = K_v \{a_v^n\}_{n \geq 0} K_v$$

and the decomposition for the Harish-Chandra function

$$\Xi(g) = \prod_{v \in V} \Xi_v(g_v)$$

where Ξ_v 's are the Harish-Chandra functions of G_v 's. Using the estimates

$$\Xi_v(a_v^n) \geq c_1 q_v^{-n/2} \quad \text{and} \quad \mathrm{vol}(K_v a_v^n K_v) \geq c_2 q_v^n$$

with some $c_1, c_2 > 0$, we conclude that for $p > 2$,

$$\int_{G_v} \Xi_v^p dm_v \geq 1 + \sum_{n \geq 1} (c_1^p c_2) q_v^{-pn/2+n} \geq 1 + c_3 q_v^{1-p/2}$$

for some $c_3 > 0$. Since the Dedekind zeta function $\prod_v (1 - q_v^{-s})^{-1}$ has a pole at $s = 1$, it follows that the product $\prod_{v \in V_f} \int_{G_v} \Xi_v^p dm_v$ diverges when $p \leq 4$. \square

Proof of Proposition 6.3. We have

$$\Xi_G(g) = \prod_{v \in V} \Xi_{G_v}(g_v), \quad g = (g_v) \in G.$$

It is well-known that the local Harish-Chandra functions Ξ_{G_v} are in $L^{2+\varepsilon}(G_v)$ for every $\varepsilon > 0$. Since Ξ_{G_v} are all bounded by 1, it suffices to prove that for some finite $V_0 \subset V$ containing the Archimedean places, the function $\prod_{v \notin V_0} \Xi_{G_v}$ is in $L^p(G)$ for $p > 4$.

In the proof, we use the explicit description of Cartan decomposition over non-Archimedean fields, which we now briefly recall (see [Ti] for details). Since K_v can be assume to be special, $G_v = K_v \mathbf{Z}(F_v) K_v$ where \mathbf{Z} is the centraliser of a suitable F_v -split torus \mathbf{S} in \mathbf{G} . Moreover, for almost all v , \mathbf{G} is split over an unramified extension of F_v , so that $G_v = K_v \mathbf{S}(F_v) K_v$. We assume that this decomposition holds for all $v \notin V_0$. Let Π_v be the set of simple roots for $\mathbf{S}(F_v)$ and

$$S_v^+ = \{s \in \mathbf{S}(F_v) : |\chi(s)|_v \geq 1 \text{ for } \chi \in \Pi_v\}.$$

Then we also have $G_v = K_v S_v^+ K_v$. We will use the following basic bound for the Harish-Chandra function:

$$(6.3) \quad \Xi_{G_v}(a_v) \leq c_\varepsilon \Delta_{P_v}(a_v)^{-1/2+\varepsilon}, \quad a_v \in S_v^+, \varepsilon > 0,$$

where Δ_{P_v} is the modular function of the group P_v (see [Si, Thm. 4.2.1]). It is clear from the proof in [Si] that the constant $c_\varepsilon > 0$ can be chosen to be bounded uniformly in v . We have

$$\Delta_{P_v}(a_v) = \prod_{\chi \in \Pi_v} |\chi(a_v)|_v^{n_{\chi,v}}$$

for some strictly positive integers $n_{\chi,v} \in \mathbb{N}$. The Haar measure of the double coset $K_v a_v K_v$ (subject to the usual normalisation $m_{G_v}(K_v) = 1$ for $v \in V_f$) satisfies the bound

$$(6.4) \quad m_{G_v}(K_v a_v K_v) \leq c \Delta_{P_v}(a_v), \quad \text{where } c > 0 \text{ is independent of } v.$$

Indeed, the estimate follows from the elementary proof of [Si, Theorem 4.1.1], which also makes it plain that the constant c is independent of v .

To prove the main estimate we combine (6.3) and (6.4), and for $p > 4$ we obtain:

$$\begin{aligned} \int_{G_v} \Xi_{G_v}(g_v)^p dm_v(g_v) &= \sum_{a_v \in K_v \backslash G_v / K_v} \Xi_{G_v}(a_v)^p m_{G_v}(K_v a_v K_v) \\ &\leq 1 + \sum_{a_v \in K_v \backslash G_v / K_v - K_v} (cc_\varepsilon^p) \Delta_{P_v}(a_v)^{p(-1/2+\varepsilon)+1} \\ &\leq 1 + \sum_{a_v \in K_v \backslash G_v / K_v - K_v} (cc_\varepsilon^p) \left(\prod_{\chi \in \Pi_v} |\chi(a_v)|_v \right)^{p(-1/2+\varepsilon)+1} \\ &\leq 1 + \sum_{i_1, \dots, i_r \in \mathbb{Z}_+, (i_1, \dots, i_r) \neq 0} (cc_\varepsilon^p) q_v^{(p(-1/2+\varepsilon)+1) \sum_{j=1}^r i_j} \\ &= 1 + O_\varepsilon \left(q_v^{p(-1/2+\varepsilon)+1} \right). \end{aligned}$$

Since the Dedekind zeta function converges absolutely for $s > 1$, it also follows that $\sum_{v \in V_f} q_v^{-s} < \infty$. Hence, $\prod_{v \notin V_0} \int_{G_v} \Xi_{G_v}^p dm_v < \infty$, as required. \square

Remark 6.5. Regarding the estimate (6.4), we note that an exact formula for the measure of a double coset was established for split simply connected groups in [Gr] as part of the discussion of the Satake transform. The fact that we may take $c = 1 + \frac{c_1}{q_v}$ is established in [STBT2, Lemma 6.11] for adjoint groups, but only as a consequence of the computation of the integral of the local height function, which is less elementary.

6.3. Analogue of the radial Kunze-Stein phenomenon on adèle groups. To complete the proof of Theorem 6.1 we need, according to our general recipe from §3.1, to prove a stable quantitative mean ergodic theorem for the Haar-uniform averages supported on the sets B_T . Our first step towards this goal is to establish a version the radial Kunze-Stein inequality for adèle group, which is of considerable independent interest.

Theorem 6.6. *Let G be as in Theorem 6.1, and $G = G(\mathbb{A})$. Let $q \in [1, 4/3)$. Then for every absolutely continuous bi- K -invariant probability measure β such that $\|\beta\|_q < \infty$ and $f \in L^2(G)$,*

$$\|\beta * f\|_2 \leq C_q \|\beta\|_q \|f\|_2.$$

Proof. Using that $G = G(\mathbb{A})$ has an Iwasawa decomposition, we can utilise Herz' argument as presented by Cowling [Co3]. The only difference is that the Harish-Chandra function $\Xi_{G(\mathbb{A})}$ is not in $L^{2+\varepsilon}(G(\mathbb{A}))$, but in $L^{4+\varepsilon}(G(\mathbb{A}))$, $\varepsilon > 0$ (see Proposition 6.3). Therefore, this argument works only for $q < 4/3$, which is the exponent dual to 4. \square

We can now establish a mean ergodic theorem with a rate for adèle groups, as follows.

Corollary 6.7. *Let G be as in Theorem 6.1, $G = G(\mathbb{A})$, and B_T be the balls w.r.t. the height function. Let G act on a standard Borel probability space (X, μ) and assume that the representation π_X^0 of G on $L^2_0(X)$ is L^{p+} for some $0 < p < \infty$, or more generally, that $(\pi_X^0)^{\otimes n_\varepsilon}$ is weakly contained in λ_G . Then the stable quantitative mean ergodic theorem holds in $L^2(X)$ for the Haar-uniform averages β_T with the following estimate:*

$$\left\| \pi_X(\beta_T)f - \int_X f d\mu \right\|_{L^2(X)} \leq C'_\eta m_G(B_T)^{-(4n_\varepsilon(p))^{-1} + \eta} \|f\|_{L^2(X)}, \quad \eta > 0.$$

Proof. As in the proof of Theorem 4.5, $(\pi_X^0)^{\otimes n_\varepsilon}$ is an L^{2+} -representation, and

$$\|\pi_X^0(\beta_T)\| \leq \|\lambda_G(\beta_T)\|^{1/n_\varepsilon}$$

where λ_G denotes the regular representation of G .

Let $\tilde{B}_T = KB_TK$ and $\tilde{\beta}_T$ denote the uniform averages supported on \tilde{B}_T . By Theorem 6.6,

$$\|\tilde{\beta}_T * f\|_2 \leq C_q m_G(\tilde{B}_T)^{-(1-1/q)} \|f\|_2.$$

for every $q \in [1, 4/3)$ and $f \in L^2(G)$. This implies that

$$\|\lambda_G(\tilde{\beta}_T)\| \leq C'_\eta m_G(\tilde{B}_T)^{-1/4+\eta}, \quad \eta > 0.$$

Since $\tilde{B}_T \subset B_{cT}$ for some $c > 0$, it follows from the volume estimate in [GMO, Section 4.3] that $m_G(\tilde{B}_T) \leq C m_G(B_T)$. Hence,

$$\|\lambda_G(\beta_T)\| \leq C \|\lambda_G(\tilde{\beta}_T)\|,$$

and the claim follows. \square

Proof of Theorem 6.1. Every irreducible unitary representation π of G is unitarily equivalent to a restricted tensor product $\pi = \bigotimes_v (\pi_v, \psi_v)$. Here π_v is an irreducible unitary representation of the local group G_v (see [F]), and for almost all $v \in V$, the representation π_v is spherical, namely the space of $G(\mathcal{O}_v)$ -invariant vectors has dimension one, with ψ_v denoting a unit vector invariant under $G(\mathcal{O}_v)$. This follows from [Mo, Lemma 6.3], the fact that π_v is irreducible, and the fact that $(G(F_v), G(\mathcal{O}_v))$ is a Gelfand pair (see also [BC, p. 733]).

It follows from the definition of the restricted tensor product (π_v, ψ_v) w.r.t. to the $G(\mathcal{O}_v)$ -invariant unit vectors ψ_v (see e.g. [Mo]), that there exists a canonical injective equivariant unitary map

$$\left(\bigotimes'_v (\pi_v, \psi_v) \right)^{\otimes n} \longrightarrow \bigotimes'_v (\pi_v^{\otimes n}, \psi_v^{\otimes n}).$$

According to [Cl], if π is weakly contained in $L_0^2(G(\mathbb{A})/G(F))$, then the local constituents π_v are L^p -representations for some uniform p independent of v . Then for $n \geq p/2$ we have $\pi_v^{\otimes n} \subset \infty \cdot \lambda_{G_v}$, and in particular $\pi_v^{\otimes n}$ is weakly contained in the regular representation λ_{G_v} . Now according to [BC, Thm. 2], if σ_v is irreducible and weakly contained in λ_{G_v} for each v , then $\bigotimes'_v (\sigma_v, \phi_v)$ (ϕ_v a $G(\mathcal{O}_v)$ -invariant unit vector) is weakly contained in $\lambda_{G(\mathbb{A})}$. The proof of this fact however makes no use of the irreducibility assumption, and so is valid for $\sigma_v = \pi_v^{\otimes n}$, $\phi_v = \psi_v^{\otimes n}$ as well. Hence

$$\pi^{\otimes n} \cong \left(\bigotimes'_v \pi_v \right)^{\otimes n} \subseteq \bigotimes'_v \pi_v^{\otimes n} \preceq \lambda_{G(\mathbb{A})},$$

Since this argument applies to every irreducible representation weakly contained in $L_0^2(G(\mathbb{A})/G(F))$, it follows that the n -th tensor power of $L_0^2(G(\mathbb{A})/G(F))$ is weakly contained in $\lambda_{G(\mathbb{A})/G(F)}$ as well.

To complete the proof of Theorem 6.1, we set $\mathcal{O}_\varepsilon = \mathcal{O}_\varepsilon^\infty \times W$ where

$$\mathcal{O}_\varepsilon^\infty = \{g = (g_v) \in G_\infty : H_v(g_v^{\pm 1} - id) < \varepsilon, v \in V_\infty\},$$

and $W \subset G_f$ is a compact open subgroup such that the height H is W -bi-invariant. Then \mathcal{O}_ε has local dimension at most $\dim(G_\infty)$. The family $\{B_{e^t}\}$ is Hölder well-rounded with respect to the neighbourhoods \mathcal{O}_ε (see [GO2, Proposition 2.19(2)]). By Theorem 6.7, the stable quantitative mean ergodic theorem holds for the action of G on $L_0^2(G(\mathbb{A})/G(F))$. Therefore Theorem 6.1 follows from Theorem 1.9. \square

7. ANGULAR DISTRIBUTION IN SYMMETRIC SPACES

7.1. Definitions, notations, and statements of results. Let G be a (non-compact) connected semisimple Lie group with finite center. Consider the Cartan decomposition

$$G = KA^+K$$

where K is a maximal compact subgroup in G , and A^+ is a closed positive Weyl chamber in a Cartan subgroup A compatible with K . The A^+ -component in this decomposition is unique, and the K -components of regular elements are unique modulo M where M is the centraliser of A in K . An element $g = k_1 a k_2$ is called δ -regular (for some $\delta > 0$) if the distance of a from the walls of the Weyl chamber is at least δ . Otherwise, the element is called δ -singular.

We denote by d the Cartan-Killing metric on the symmetric space G/K and set $D_t = \{g \in G : d(gK, K) \leq t\}$. For $\Phi, \Psi \subset K$, we consider bisectors:

$$D_t(\Phi, \Psi) = \{k_1 a k_2 : k_1 \in \Phi, a \in A^+, d(aK, K) \leq t, k_2 \in \Psi\}.$$

We are interested in the distribution of lattice points with respect to bisectors. The main term in this problem was investigated in [GO1] and [GOS2], but the issue of rates in the asymptotic estimates was not addressed.

We denote by D_t^δ the subset of δ -singular elements of D_t . It will be crucial that most of the volume is concentrated in the interior of the Weyl chamber, i.e., there exists $\zeta_0 > 0$ such that for every $\delta > 0$,

$$(7.1) \quad \text{vol}(D_t^\delta) = O_{\delta,\eta}(\text{vol}(D_t)^{1-\zeta_0+\eta}), \quad \eta > 0,$$

Let us note that (7.7) below gives the precise value of ζ_0 in terms of the root system of G . The lower order of magnitude of the volume of the neighbourhood D_t^δ of the singular set is the main difference between the bisectors on Riemannian symmetric spaces discussed in the present section, and bisectors in more general affine symmetric spaces considered in Section 8.

We fix a base of neighbourhoods \mathcal{O}_ε of identity in G with respect to a (right) invariant Riemannian metric. A measurable subset Φ of a homogeneous space of K is called Lipschitz well-rounded if

$$\text{vol}((\mathcal{O}_\varepsilon \cap K)\Phi - \cap_{u \in \mathcal{O}_\varepsilon \cap K} u\Phi) \ll_\Phi \varepsilon$$

for $\varepsilon \in (0, \varepsilon_1)$. For example, it is easy to check that balls with respect to an invariant Riemannian metric are Lipschitz well-rounded. Note that this notion does not depend on a choice of a Riemannian metric.

Theorem 7.1. *Let Γ be a lattice in G such that the representation $\pi_{G/\Gamma}^0$ in $L_0^2(G/\Gamma)$ is L^{p+} for some $p > 0$, Φ a Lipschitz well-rounded subset of K/M with positive measure, and Ψ a Lipschitz well-rounded subset of $M \setminus K$ with positive measure. Then*

$$|\Gamma \cap D_t(\Phi, \Psi)| = \frac{\text{vol}(D_t(\Phi, \Psi))}{\text{vol}(G/\Gamma)} + O_{\Phi, \Psi, \eta}(\text{vol}(D_t)^{1-\zeta+\eta}), \quad \eta > 0,$$

where $\zeta = \min\{\zeta_0, (2n_e(p))^{-1}(1 + \dim G)^{-1}\}$. Moreover, this estimate is uniform over all lattices such that the representation $L_0^2(G/\Gamma)$ is L^{p+} and $\varepsilon_0(e, \Gamma) \geq \varepsilon_0$ with fixed $\varepsilon_0 > 0$.

We also state a version of this theorem in the language of test-functions. Since it is essentially equivalent to Theorem 7.1, we only give a proof of Theorem 7.1. For an element $g \in G$, we write its Cartan decomposition as $g = k_1(g)a(g)k_2(g)$.

Theorem 7.2. *Let Γ be a lattice in G such that the representation $\pi_{G/\Gamma}^0$ in $L_0^2(G/\Gamma)$ is L^{p+} for some $p > 0$, ϕ_1 a Lipschitz function on K/M , and ϕ_2 a Lipschitz function on $M \setminus K$. Then*

$$\begin{aligned} \sum_{\gamma \in \Gamma \cap D_t} \phi_1(k_1(\gamma))\phi_2(k_2(\gamma)) &= \frac{\text{vol}(D_t)}{\text{vol}(G/\Gamma)} \left(\int_{K/M} \phi_1 dk \right) \left(\int_{M \setminus K} \phi_2 dk \right) \\ &+ O_{\phi_1, \phi_2, \eta}(\text{vol}(D_t)^{1-\zeta+\eta}), \quad \eta > 0, \end{aligned}$$

where ζ is as in Theorem 7.1.

The proof of Theorem 7.1 is based on Theorem 1.9. According to the recipe of §3.1, we must verify the spectral condition for the averages supported on $D_t(\Phi, \Psi)$, and then show that they are Hölder well-rounded. The spectral estimate is an immediate consequence of Theorem 4.5. The main issue here is the regularity of the sets, and we now undertake the task of showing that they are in fact Lipschitz well-rounded.

7.2. Quantitative wave front lemma. It was shown in [N3] that the components of the Cartan decomposition $G = KA^+K$ of regular elements vary continuously under small perturbations. The proof is very simple and based on the proof of the wave-front Lemma given in [EMM, Lemma 5.11] (see also [EM, Theorem 3.1]). We apply this argument to show that the variation of the Cartan components is Lipschitz. See also [GOS2] for a different argument.

For $\delta > 0$, we denote by \tilde{A}^δ the subset of A^+ consisting of elements with distance $\geq \delta$ from the walls.

Proposition 7.3. Effective Cartan decomposition. *Let $\delta > 0$. There exist $\varepsilon_0, \ell_0 > 0$ such that for every $g = k_1 a k_2 \in K \tilde{A}^\delta K$ and $\varepsilon \in (0, \varepsilon_0)$,*

$$\mathcal{O}_\varepsilon g \mathcal{O}_\varepsilon \subset (\mathcal{O}_{\ell_0 \varepsilon} \cap K) k_1 M (\mathcal{O}_{\ell_0 \varepsilon} \cap A) a k_2 (\mathcal{O}_{\ell_0 \varepsilon} \cap K).$$

Proof. Using that K is compact, it is easy to reduce the proof to showing that

$$a \mathcal{O}_\varepsilon \subset (\mathcal{O}_{\ell_0 \varepsilon} \cap K) M (\mathcal{O}_{\ell_0 \varepsilon} \cap A) a (\mathcal{O}_{\ell_0 \varepsilon} \cap K).$$

Since the proof of [EMM, Lemma 5.11] implies that

$$a \mathcal{O}_\varepsilon \subset K (\mathcal{O}_{\ell_0 \varepsilon} \cap A) a K$$

for some $\ell_0 > 0$, it remains to analyse behaviour of the K -components.

Let P be the standard parabolic subgroup, i.e., $P = MAU$ where M is the centraliser of A in K and U is the subgroup generated by positive root subgroups. There exists a real representation $G \rightarrow \mathrm{GL}(V)$ such that for some vector $e_1 \in V$, its projective stabiliser is P [GJT, Theorem 4.29]. Without loss of generality, we may assume that $V = \langle Ge_1 \rangle$. One can choose a Euclidean structure on V such that K consists of orthogonal matrices, A consists of self-adjoint matrices, and $\|e_1\| = 1$. We fix an orthonormal basis $\{e_i\}$ of eigenvectors of A . If $ae_1 = e^{\lambda(\log a)} e_1$ for $\lambda \in \mathrm{Lie}(A)^*$, then the weights of e_i 's are of the form $\lambda - \alpha_i$ where α_i is a positive linear combination of positive roots. In particular, it follows that for every $a \in A^+$, $\|a\| = e^{\lambda(\log a)}$.

Let $a \in \tilde{A}^\delta$, $g \in \mathcal{O}_\varepsilon$ and $ag = k_1 b k_2$ for $k_1, k_2 \in K$ and $b \in A^+$. We will show that for some $c > 0$, we have $\|k_1 e_1 - e_1\| < c\varepsilon$ and $\|k_2^{-1} e_1 - e_1\| < c\varepsilon$. Since $P \cap K = M$, this implies that $k_1 \in (\mathcal{O}_{c'\varepsilon} \cap K) M$ and $k_2 \in M (\mathcal{O}_{c'\varepsilon} \cap K)$ for some $c' > 0$, as required.

We have

$$(7.2) \quad e^{\lambda(\log b)} = \langle b e_1, e_1 \rangle = \langle k_1^{-1} a g k_2^{-1} e_1, e_1 \rangle \leq \|a g k_2^{-1} e_1\|.$$

Writing $g k_2^{-1} e_1 = \sum_i u_i e_i$ with $u_i \in \mathbb{R}$, we get

$$\|a g k_2^{-1} e_1\|^2 = e^{2\lambda(\log a)} \sum_i e^{-2\alpha_i(\log a)} u_i^2 \leq e^{2\lambda(\log a)} \left(u_1^2 + \sum_{i>1} e^{-c_1 \delta} u_i^2 \right)$$

for some $c_1 > 0$. Since $g \in \mathcal{O}_\varepsilon$,

$$(7.3) \quad \|g k_2^{-1} e_1\|^2 \leq 1 + c_2 \varepsilon$$

for some $c_2 > 0$, and

$$(7.4) \quad \|a g k_2^{-1} e_1\|^2 \leq e^{2\lambda(\log a)} (u_1^2 + e^{-c_1 \delta} (1 + c_2 \varepsilon - u_1^2)).$$

Since $\|b\|^2 = \|a g\|^2 \geq (1 - c_3 \varepsilon) \|a\|^2$ for some $c_3 > 0$, it follows that $e^{2\lambda(\log b) - 2\lambda(\log a)} \geq 1 - c_3 \varepsilon$. Hence, combining (7.2) and (7.4), we get

$$u_1^2 + e^{-c_1 \delta} (1 + c_2 \varepsilon - u_1^2) \geq 1 - c_3 \varepsilon,$$

and

$$u_1^2 \geq \frac{1 - c_3\varepsilon - e^{-c_1\delta}(1 + c_2\varepsilon)}{1 - e^{-c_1\delta}}.$$

This shows that $|u_1| = 1 + O_\delta(\varepsilon)$. Then by (7.3), $\|gk_2^{-1}e_1 - e_1\| = O_\delta(\varepsilon)$. Since $g \in \mathcal{O}_\varepsilon$, it follows that $\|k_2^{-1}e_1 - e_1\| = O_\delta(\varepsilon)$ as well.

The proof that $\|k_1e_1 - e_1\| = O_\delta(\varepsilon)$ is similar. \square

Proposition 7.4. *Let Φ be a Lipschitz well-rounded subset of K/M with positive measure and Ψ a Lipschitz well-rounded subset of $M \setminus K$ with positive measure. Then for every $\delta > 0$, the family of sets*

$$\tilde{D}_t^\delta(\Phi, \Psi) := \{k_1ak_2 : k_1 \in \Phi, a \in \tilde{A}^\delta, d(aK, K) \leq t, k_2 \in \Psi\}.$$

is Lipschitz well-rounded.

Before we start the proof, we recall some facts about volumes. The Haar measure in KA^+K -coordinates is given by $dk_1 \xi(a) da dk_2$ where dk_1, da, dk_2 are Haar measures on the components and

$$(7.5) \quad \xi(a) = \prod_{\alpha \in \Sigma^+} (e^{\alpha(\log a)} - e^{-\alpha(\log a)})$$

(here Σ^+ is the set of positive roots). We denote by 2ρ the sum of positive roots with multiplicities. It is well-known that for every $\eta > 0$ and $c_\eta > 1$,

$$(7.6) \quad c_\eta^{-1} e^{(\alpha-\eta)t} \leq \text{vol}(D_t) \leq c_\eta e^{(\alpha+\eta)t}$$

where $\alpha = \max\{2\rho(\log a) : a \in A^+ \cap D_1\}$.¹ We also set $\alpha_0 = \max\{2\rho(\log a) : a \in \text{walls}(A^+) \cap D_1\}$. Since the balls are strictly convex, $\alpha_0 < \alpha$. Hence, using that $\xi(a) \leq e^{2\rho(\log a)}$, we deduce from (7.6) that

$$(7.7) \quad \text{vol}(D_t^\delta) = O_{\delta,\eta} \left(\text{vol}(D_t)^{\alpha_0/\alpha+\eta} \right), \quad \eta > 0.$$

Proof of Proposition 7.4. By Proposition 7.3,

$$\mathcal{O}_\varepsilon \tilde{D}_t^\delta(\Phi, \Psi) \mathcal{O}_\varepsilon \subset \tilde{D}_{t+2\varepsilon}^{\delta-\ell_0\varepsilon}(\Phi_\varepsilon^+, \Psi_\varepsilon^+) \quad \text{and} \quad \bigcap_{u,v \in \mathcal{O}_\varepsilon} u \tilde{D}_t^\delta(\Phi, \Psi) v \supset \tilde{D}_{t-2\varepsilon}^{\delta+\ell_0\varepsilon}(\Phi_\varepsilon^-, \Psi_\varepsilon^-)$$

where $\Phi_\varepsilon^+ = (\mathcal{O}_\varepsilon \cap K)\Phi$, $\Psi_\varepsilon^+ = \Psi(\mathcal{O}_\varepsilon \cap K)$, $\Phi_\varepsilon^- = \bigcap_{u \in \mathcal{O}_\varepsilon \cap K} u\Phi$, $\Psi_\varepsilon^- = \bigcap_{u \in \mathcal{O}_\varepsilon \cap K} \Psi u$. Hence, it remains to estimate

$$\begin{aligned} & \text{vol} \left(\tilde{D}_{t+2\varepsilon}^{\delta-\ell_0\varepsilon}(\Phi_\varepsilon^+, \Psi_\varepsilon^+) - \tilde{D}_{t-2\varepsilon}^{\delta+\ell_0\varepsilon}(\Phi_\varepsilon^-, \Psi_\varepsilon^-) \right) \\ & \leq \text{vol} \left(\tilde{D}_{t+2\varepsilon}^{\delta-\ell_0\varepsilon}(\Phi_\varepsilon^+ - \Phi_\varepsilon^-, \Psi_\varepsilon^+) \right) + \text{vol} \left(\tilde{D}_{t+2\varepsilon}^{\delta-\ell_0\varepsilon}(\Phi_\varepsilon^+, \Psi_\varepsilon^+ - \Psi_\varepsilon^-) \right) \\ & \quad + \text{vol} \left(\tilde{D}_{t+2\varepsilon}^{\delta-\ell_0\varepsilon} - \tilde{D}_{t-2\varepsilon}^{\delta+\ell_0\varepsilon} \right). \end{aligned}$$

Since the sets Φ and Ψ are Lipschitz well-rounded, the first and the second terms are $O(\varepsilon \text{vol}(D_{t+2\varepsilon}))$. We estimate the last term by

$$\text{vol} \left(\tilde{D}_{t+2\varepsilon}^{\delta-\ell_0\varepsilon} - \tilde{D}_{t-2\varepsilon}^{\delta+\ell_0\varepsilon} \right) \leq \text{vol}(D_{t+2\varepsilon} - D_{t-2\varepsilon}) + \text{vol} \left(\tilde{D}_{t+2\varepsilon}^{\delta-\ell_0\varepsilon} - \tilde{D}_{t+2\varepsilon}^{\delta+\ell_0\varepsilon} \right).$$

¹In fact, the exact asymptotic is known, but we do not need it here.

It was shown in [GN, Proposition 7.1] that the function $t \mapsto \log \text{vol}(D_t)$ is uniformly locally Lipschitz. It follows from the formula for the Haar measure, (7.5) and (7.6) that

$$\text{vol}\left(\tilde{D}_{t+2\varepsilon}^{\delta-\ell_0\varepsilon} - \tilde{D}_{t+2\varepsilon}^{\delta+\ell_0\varepsilon}\right) \ll \varepsilon t^{\dim A^+ - 1} e^{\alpha_0 t} \ll \varepsilon \text{vol}(D_{t+2\varepsilon}).$$

We conclude that

$$\text{vol}\left(\tilde{D}_{t+2\varepsilon}^{\delta-\ell_0\varepsilon}(\Phi_\varepsilon^+, \Psi_\varepsilon^+) - \tilde{D}_{t-2\varepsilon}^{\delta+\ell_0\varepsilon}(\Phi_\varepsilon^-, \Psi_\varepsilon^-)\right) = O(\varepsilon \text{vol}(D_{t+2\varepsilon})).$$

Finally, the claim follows from (7.7) and the Lipschitz property of the function $t \mapsto \log \text{vol}(D_t)$. \square

Proof of Theorem 7.1. By Proposition 7.4, the family $\{\tilde{D}_t^\delta(\Phi, \Psi)\}$ is Lipschitz well-rounded, and by Theorem 4.5, the corresponding averages satisfy the stable quantitative mean ergodic theorem. Hence, by Theorem 1.9,

$$|\Gamma \cap \tilde{D}_t^\delta(\Phi, \Psi)| = \frac{\text{vol}(\tilde{D}_t^\delta(\Phi, \Psi))}{\text{vol}(G/\Gamma)} + O_{\Phi, \Psi, \eta}\left(\text{vol}(\tilde{D}_t^\delta(\Phi, \Psi))^{1-\zeta_1+\eta}\right), \quad \eta > 0,$$

where $\zeta_1 = (2n_e(p))^{-1}(1 + \dim G)^{-1}$. Hence, by (7.7),

$$(7.8) \quad |\Gamma \cap \tilde{D}_t^\delta(\Phi, \Psi)| = \frac{\text{vol}(\tilde{D}_t^\delta(\Phi, \Psi))}{\text{vol}(G/\Gamma)} + O_{\Phi, \Psi, \eta}\left(\text{vol}(D_t)^{1-\zeta_1+\eta}\right), \quad \eta > 0.$$

It remains to estimate the number of lattice points in the singular set, namely $|\Gamma \cap D_t^\delta(\Phi, \Psi)|$. To that end, fix a symmetric neighbourhood \mathcal{O}_ω of identity such that $\Gamma \cap \mathcal{O}_\omega^2 = \{e\}$. Then

$$|\Gamma \cap D_t^\delta(\Phi, \Psi)| \leq \frac{\text{vol}(\mathcal{O}_\omega D_t^\delta)}{\text{vol}(\mathcal{O}_\omega)},$$

and by Proposition 7.3,

$$\text{vol}(\mathcal{O}_\omega D_t^\delta) \leq \text{vol}(D_{t+\omega}^{\delta-\ell_0\omega}).$$

Hence, by (7.1) and the Lipschitz property of the function $t \mapsto \log \text{vol}(D_t)$,

$$(7.9) \quad |\Gamma \cap D_t^\delta(\Phi, \Psi)| = O_{\delta, \eta}\left(\text{vol}(D_t)^{1-\zeta_0+\eta}\right).$$

Finally, combining (7.8) and (7.9), we deduce the claim. \square

8. LATTICE POINTS ON AFFINE SYMMETRIC VARIETIES

In the present section we consider the lattice point counting problem in subsets of a connected semisimple Lie group arising from sectors in affine symmetric spaces, and give an explicit quantitative estimate of the error in all cases. This result gives an explicit quantitative solution of the lattice point counting problem on G/H itself whenever $\Gamma \cap H$ is co-compact in H .

We note also that our solution will be uniform over all subgroups of finite index in the lattice, provided they all admit a uniform spectral gap, namely satisfy property τ . This uniformity property plays a crucial role in establishing the existence of the right order of magnitude of almost prime points on the algebraic variety G/H , provided G and H are defined over \mathbb{Q} and Γ is the lattice of integral points. This and other applications of the solution of the lattice point counting problem will be elaborated elsewhere.

8.1. Notation, definitions and statements of results. Throughout the present section, we let G be a connected semisimple Lie group with finite center and H is a closed symmetric subgroup of G (that is, the Lie algebra of H is the set of fixed points of an involution σ). Let K be a maximal compact subgroup compatible with H (this means that the involution θ corresponding to K commutes with σ). Let $G^{\sigma\theta}$ be the subgroup of fixed points of $\sigma\theta$ and A a Cartan subgroup of $G^{\sigma\theta}$ compatible with $K \cap G^{\sigma\theta}$. The group A is equipped with a root system (for the action of A on $G^{\sigma\theta}$). We fix a system of positive roots and denote by A^+ a closed positive Weyl chamber in A . Then we have the Cartan decomposition

$$G = KA^+H.$$

We say that an element $a \in A$ is δ -regular if the distance of a from the boundary of A^+ is at least δ , and regular if it δ -regular for some $\delta > 0$. More generally, an element $g \in G$ is called δ -regular if its A -component is δ -regular. Note that the A^+ -component of an element is uniquely defined, and the K - and H -components of a regular element are uniquely defined modulo the subgroup M which is the centraliser of A in $K \cap H$. We refer to [Sc, Ch. 7] and [HS, Part II] for basic facts about affine symmetric spaces.

Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a representation of G , and $v_0 \in V$ be such that $\mathrm{Stab}_G(v_0) = H$. We fix a norm on V and define

$$S_t = \{g \in G : \log \|gv_0\| \leq t\} \quad \text{and} \quad B_t = \{g \in G/H : \log \|gv_0\| \leq t\}.$$

For sets $\Phi \subset K$ and $\Psi \subset H$, define

$$S_t(\Phi, \Psi) = S_t \cap \Phi A^+ \Psi.$$

We compute the asymptotics of the number of lattice points in $S_t(\Phi, \Psi)$. Our argument is based on the effective version of the Cartan decomposition, which draws on some arguments in [GOS2]. The main term in the asymptotic of lattice points $S_t(\Phi, \Psi)$ was also computed in [GOS2], but the problem of rates was not addressed. As noted in §1.3, the method of mixing used there generally gives an error term inferior to the one established below.

Theorem 8.1. *Let Γ be a lattice in G such that the representation $\pi_{G/\Gamma}^0$ in $L_0^2(G/\Gamma)$ is L^{p+} for some $p > 0$. Let Φ be Lipschitz well-rounded subset of K/M with positive measure, and Ψ a bounded Lipschitz well-rounded subset of $M \backslash H$ with positive measure. Then*

$$|\Gamma \cap S_t(\Phi, \Psi)| = \frac{\mathrm{vol}(S_t(\Phi, \Psi))}{\mathrm{vol}(G/\Gamma)} + O_{\Phi, \Psi, \eta}(\mathrm{vol}(S_t(\Phi, \Psi))^{1-\zeta+\eta}), \quad \eta > 0,$$

where $\zeta = (2n_e(p))^{-1}(1 + 3 \dim G)^{-1}$. Moreover, this estimate is uniform over all lattices such that the representation $L_0^2(G/\Gamma)$ is L^{p+} and $\varepsilon_0(e, \Gamma) \geq \varepsilon_0$ with fixed $\varepsilon_0 > 0$.

Remark 8.2. Let us comment on the difference between Theorem 7.1 and Theorem 8.3. While the balls on symmetric spaces are defined with respect to the Cartan-Killing metric, the ball in affine symmetric spaces are defined with respect to a norm. In the later case, an analogue of estimate (7.1) fails, and we will need a more elaborate argument to prove well-roundedness. As a result, while the family $D_t(\Phi, \Psi)$ in Riemannian symmetric spaces is shown to be Lipschitz well-rounded, we can only show that the family $S_t(\Phi, \Psi)$ in affine symmetric spaces is Hölder well-rounded with exponent $1/3$.

Let Γ be a lattice in G such that $\Gamma \cap H$ is cocompact in H . Then the orbit Γv_0 is discrete. Given $\Phi \subset K/M$, we are interested in the effective asymptotic of

$$\Gamma v_0 \cap \{v \in \Phi A^+ v_0 : \log \|v\| \leq t\}.$$

Corollary 8.3. *Let Φ be a Lipschitz well-rounded subset of K/M with positive measure. Then for $B_t(\Phi) = \Phi A^+ H \cap B_t$,*

$$|\Gamma H \cap B_t(\Phi)| = \frac{\text{vol}(H/(H \cap \Gamma))}{\text{vol}(G/\Gamma)} \text{vol}(B_t(\Phi)) + O_{\Phi, \eta}(\text{vol}(B_t(\Phi))^{1-\zeta+\eta}), \quad \eta > 0,$$

where ζ is as in Theorem 8.1.

As noted already, the crucial step in the proof is to show that the family of sets $S_t(\Phi, \Psi)$ is Hölder well-rounded.

Proposition 8.4. *Let Φ be a Lipschitz well-rounded subset of K/M with positive measure and Ψ a bounded Lipschitz well-rounded subset of $M \setminus H$ with positive measure. Then the family of sets $S_t(\Phi, \Psi)$ is Hölder well-rounded with exponent $1/3$.*

Remark 8.5. More generally, it will be clear from the proof that any family of measurable subsets of $S_t(\Phi, \Psi)$ that contain all regular elements of $S_t(\Phi, \Psi)$ is Hölder well-rounded with exponent $1/3$. This remark will be used below.

8.2. Hölder well-roundedness of sector averages. In preparation for the proof of Proposition 8.4, we note the following quantitative result on the Lipschitz property of the Cartan decomposition, which is based on arguments appearing in [GOS2]. Let \mathcal{O}_ε denote the ε -neighbourhood of identity in G with respect to a (right) invariant Riemannian metric in G , so that $\mathcal{O}_{\varepsilon_1} \cdot \mathcal{O}_{\varepsilon_2} = \mathcal{O}_{\varepsilon_1 + \varepsilon_2}$.

Proposition 8.6. *Let $\delta \in (0, 1)$. There exist $\varepsilon_0, \ell_0 > 0$ such that for every δ -regular $a \in A$ and $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\mathcal{O}_\varepsilon a \mathcal{O}_\varepsilon \subset (\mathcal{O}_{\ell_0 \varepsilon} \cap K) (\mathcal{O}_{\ell_0 \varepsilon} \cap A) a (\mathcal{O}_{\ell_0 \varepsilon} \cap H).$$

Moreover,

$$\varepsilon_0 \gg \delta^3 \text{ and } \ell_0 \ll \delta^{-2} \text{ as } \delta \rightarrow 0^+.$$

Proof. It was shown in [GOS2, Theorem 4.1], that there exist $\varepsilon_0, \ell_0 > 0$ such that for every δ -regular $a \in A$ and $\varepsilon \in (0, \varepsilon_0)$,

$$\mathcal{O}_\varepsilon a \subset (\mathcal{O}_{\ell_0 \varepsilon} \cap K) (\mathcal{O}_{\ell_0 \varepsilon} \cap A) a (\mathcal{O}_{\ell_0 \varepsilon} \cap H).$$

The estimates $\varepsilon_0 \gg \delta^2$ and $\ell_0 \ll \delta^{-1}$ as $\delta \rightarrow 0^+$ can be extracted from the proof.

A straight-forward modification of the arguments in the proof in [GOS2, Theorem 4.1] also gives that

$$a \mathcal{O}_\varepsilon \subset (\mathcal{O}_{\ell_0 \varepsilon} \cap K) (\mathcal{O}_{\ell_0 \varepsilon} \cap A) a (\mathcal{O}_{\ell_0 \varepsilon} \cap H).$$

There exists $c > 0$ such that for every $\varepsilon \in (0, \varepsilon_0/(2\ell_0 + 1))$,

$$\begin{aligned} \mathcal{O}_\varepsilon a \mathcal{O}_\varepsilon &\subset (\mathcal{O}_{\ell_0 \varepsilon} \cap K) (\mathcal{O}_{\ell_0 \varepsilon} \cap A) a (\mathcal{O}_{\ell_0 \varepsilon} \cap H) \cdot \mathcal{O}_\varepsilon \\ &\subset (\mathcal{O}_{\ell_0 \varepsilon} \cap K) a \mathcal{O}_{(2\ell_0+1)\varepsilon} \\ &\subset (\mathcal{O}_{\ell_0 \varepsilon} \cap K) (\mathcal{O}_{\ell_0(2\ell_0+1)\varepsilon} \cap K) (\mathcal{O}_{\ell_0(2\ell_0+1)\varepsilon} \cap A) a (\mathcal{O}_{\ell_0(2\ell_0+1)\varepsilon} \cap H) \\ &\subset (\mathcal{O}_{\ell_0(2\ell_0+2)\varepsilon} \cap K) (\mathcal{O}_{\ell_0(2\ell_0+1)\varepsilon} \cap A) a (\mathcal{O}_{\ell_0(2\ell_0+1)\varepsilon} \cap H). \end{aligned}$$

This implies the proposition. \square

Fix a positive Weyl chamber A^{++} in A for the action of A on G . Note that this Weyl chamber is smaller than A^+ , and A^+ is finite union of chambers of the form A^{++} . Let Z denote the centraliser of A in G , and U^+ and U^- are expanding and contracting subgroups corresponding to A^{++} .

Proposition 8.7. *There exist $c, \varepsilon_0 > 0$ such that for every $a \in A^{++}$ and $\varepsilon \in (0, \varepsilon_0)$,*

$$\mathcal{O}_\varepsilon a \mathcal{O}_\varepsilon \subset (U^- \cap \mathcal{O}_{c\varepsilon})(Z \cap \mathcal{O}_{c\varepsilon})a(U^+ \cap \mathcal{O}_{c\varepsilon}).$$

Proof. Since the product map $U^- \times Z \times U^+ \rightarrow G$ is a diffeomorphism in a neighbourhood of identity, there exist $c_0, \varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$,

$$\mathcal{O}_\varepsilon \subset (U^- \cap \mathcal{O}_{c_0\varepsilon})(Z \cap \mathcal{O}_{c_0\varepsilon})(U^+ \cap \mathcal{O}_{c_0\varepsilon}).$$

There exist $c_1, \varepsilon_1 > 0$ such that for every $a \in A^{++}$ and $\varepsilon \in (0, \varepsilon_1)$,

$$a^{-1}(U^+ \cap \mathcal{O}_\varepsilon)a \subset (U^+ \cap \mathcal{O}_{c_1\varepsilon}),$$

$$a(U^- \cap \mathcal{O}_\varepsilon)a^{-1} \subset (U^- \cap \mathcal{O}_{c_1\varepsilon}).$$

Hence, it follows that

$$\begin{aligned} \mathcal{O}_\varepsilon a \mathcal{O}_\varepsilon &\subset (U^- \cap \mathcal{O}_{c_0\varepsilon})a \cdot a^{-1}(Z \cap \mathcal{O}_{c_0\varepsilon})(U^+ \cap \mathcal{O}_{c_0\varepsilon})a \cdot \mathcal{O}_\varepsilon \\ &\subset (U^- \cap \mathcal{O}_{c_0\varepsilon})a \cdot \mathcal{O}_{(c_1c_0+c_0+1)\varepsilon}, \end{aligned}$$

and

$$\begin{aligned} a \cdot \mathcal{O}_\varepsilon &\subset a(U^- \cap \mathcal{O}_{c_0\varepsilon})(Z \cap \mathcal{O}_{c_0\varepsilon})a^{-1} \cdot a(U^+ \cap \mathcal{O}_{c_0\varepsilon}) \\ &\subset (U^- \cap \mathcal{O}_{c_1c_0\varepsilon})(Z \cap \mathcal{O}_{c_0\varepsilon})a(U^+ \cap \mathcal{O}_{c_0\varepsilon}). \end{aligned}$$

Now the proposition follows from the last two estimates. \square

Proof of Proposition 8.4. Let $\varepsilon, \delta, \ell > 0$ be such that for every δ -regular $a \in A$,

$$\mathcal{O}_\varepsilon a \mathcal{O}_\varepsilon \subset (\mathcal{O}_{\ell\varepsilon} \cap K)(\mathcal{O}_{\ell\varepsilon} \cap A)a(\mathcal{O}_{\ell\varepsilon} \cap H).$$

By Proposition 8.6, for every small $\varepsilon > 0$, such δ and ℓ exist, and we have

$$(8.1) \quad \delta = O(\varepsilon^{1/3}) \quad \text{and} \quad \ell = O(\varepsilon^{-2/3}).$$

Let $S_t^\delta(\Phi, \Psi)$ denote the subset of δ -singular elements of $S_t(\Phi, \Psi)$. We claim that

$$(8.2) \quad \text{vol}(\mathcal{O}_\varepsilon S_t^\delta(\Phi, \Psi) \mathcal{O}_\varepsilon) \ll \varepsilon^{1/3} \text{vol}(S_t(\Phi, \Psi)).$$

Decomposing A^+ into a union of the Weyl chambers A^{++} , it suffices to prove this estimate for a subset of $S_t^\delta(\Phi, \Psi)$ with A -component contained in A^{++} . Let A_t^δ be the subset of $S_t \cap A^{++}$ consisting of δ -singular elements. There exists $c = c(\Phi) > 0$ such that

$$S_t^\delta(\Phi, \Psi) \cap KA^{++}H \subset \Phi A_{t+c}^\delta \Psi.$$

Let $k_1, \dots, k_I \in \Phi$ be an ε -net in $\Phi \subset K/M$ such that $I = O(\varepsilon^{-(\dim K - \dim M)})$ and $h_1, \dots, h_J \in H$ an ε -net in $M\Psi$ such that $J = O(\varepsilon^{-\dim H})$. Then for some $c_1 > 0$,

$$\mathcal{O}_\varepsilon \cdot (\Phi A_{t+c}^\delta \Psi) \cdot \mathcal{O}_\varepsilon \subset \bigcup_{i,j} \mathcal{O}_{2\varepsilon} k_i A_{t+c}^\delta h_j \mathcal{O}_{2\varepsilon} \subset \bigcup_{i,j} k_i \mathcal{O}_{c_1\varepsilon} A_{t+c}^\delta \mathcal{O}_{c_1\varepsilon} h_j.$$

Hence,

$$(8.3) \quad \text{vol}(\mathcal{O}_\varepsilon \cdot (\Phi A_{t+c}^\delta \Psi) \cdot \mathcal{O}_\varepsilon) \leq IJ \text{vol}(\mathcal{O}_{c_1\varepsilon} A_{t+c}^\delta \mathcal{O}_{c_1\varepsilon}).$$

By Proposition 8.7, for some $c_2 > 0$,

$$\mathcal{O}_{c_1\varepsilon} A_{t+c}^\delta \mathcal{O}_{c_1\varepsilon} \subset (U^- \cap \mathcal{O}_{c_2\varepsilon})(Z \cap \mathcal{O}_{c_2\varepsilon})A_{t+c}^\delta(U^+ \cap \mathcal{O}_{c_2\varepsilon}).$$

Since Z is reductive and A lies in the center of Z , there exist a local Lie subgroup Z' complementary to A and for some $c_3 > 0$,

$$(Z \cap \mathcal{O}_{c_2\varepsilon}) \subset (Z' \cap \mathcal{O}_{c_3\varepsilon})(A \cap \mathcal{O}_{c_3\varepsilon}).$$

Therefore, for some $c_4 > 0$,

$$\mathcal{O}_{c_1\varepsilon} A_{t+c}^\delta \mathcal{O}_{c_1\varepsilon} \subset (U^- \cap \mathcal{O}_{c_2\varepsilon})(Z' \cap \mathcal{O}_{c_3\varepsilon}) A_{t+c_4}^{\delta+c_3\varepsilon} (U^+ \cap \mathcal{O}_{c_2\varepsilon}).$$

The Haar measure on G with respect to $U^- Z' A U^+$ -coordinates is given by

$$\det(\text{Ad}(a)|_{U^+}) du^- dz' dadu^+.$$

Therefore,

$$\text{vol}(\mathcal{O}_{c_1\varepsilon} A_{t+c}^\delta \mathcal{O}_{c_1\varepsilon}) \ll \varepsilon^{\dim U^- + \dim Z' + \dim U^+} \int_{A_{t+c_4}^{\delta+c_3\varepsilon}} \det(\text{Ad}(a)|_{U^+}) da.$$

The last integral was estimated in [GOS2, Proposition 3.22] (see also [GOS2, Proposition 3.8]). We have

$$\int_{A_{t+c_4}^{\delta+c_3\varepsilon}} \det(\text{Ad}(a)|_{U^+}) da \ll (\delta + c_3\varepsilon) \text{vol}(S_t(\Phi, \Psi)).$$

Hence,

$$\text{vol}(\mathcal{O}_{c_1\varepsilon} A_{t+c}^\delta \mathcal{O}_{c_1\varepsilon}) \ll (\delta + \varepsilon) \varepsilon^{\dim G - \dim A} \text{vol}(S_t(\Phi, \Psi)).$$

Since

$$IJ \ll \varepsilon^{-(\dim K + \dim H - \dim M)} = \varepsilon^{-(\dim G - \dim A)},$$

we deduce from (8.3) that

$$\text{vol}(\mathcal{O}_\varepsilon S_t^\delta(\Phi, \Psi) \mathcal{O}_\varepsilon) \ll (\delta + \varepsilon) \text{vol}(S_t(\Phi, \Psi)) \ll \varepsilon^{1/3} \text{vol}(S_t(\Phi, \Psi)),$$

as claimed.

Let $\tilde{S}_t^\delta(\Phi, \Psi)$ denote the subset of δ -regular elements in $S_t(\Phi, \Psi)$. Since Φ and Ψ are bounded, it follows from Proposition 8.6 that there exists $c > 0$ such that for every $kah \in \tilde{S}_t^\delta(\Phi, \Psi)$,

$$\begin{aligned} \mathcal{O}_\varepsilon kah \mathcal{O}_\varepsilon &\subset k \mathcal{O}_{c\varepsilon} a \mathcal{O}_{c\varepsilon} h \subset k(K \cap \mathcal{O}_{\ell c\varepsilon})(A \cap \mathcal{O}_{\ell c\varepsilon}) a (H \cap \mathcal{O}_{\ell c\varepsilon}) h \\ &\subset (K \cap \mathcal{O}_{\ell c^2\varepsilon}) k (A \cap \mathcal{O}_{\ell c\varepsilon}) a h (H \cap \mathcal{O}_{\ell c^2\varepsilon}). \end{aligned}$$

Using that for some $c_1 > 0$ we have $\mathcal{O}_\varepsilon S_t \subset S_{t+c_1\varepsilon}$, we deduce from the previous estimate that

$$\mathcal{O}_\varepsilon \tilde{S}_t^\delta(\Phi, \Psi) \mathcal{O}_\varepsilon \subset \tilde{S}_{t+2c_1\ell c^2\varepsilon}^{\delta-\ell c\varepsilon}((K \cap \mathcal{O}_{\ell c^2\varepsilon})\Phi, \Psi(H \cap \mathcal{O}_{\ell c^2\varepsilon})).$$

Then by the uniqueness properties of the Cartan decomposition,

$$(8.4) \quad \begin{aligned} \mathcal{O}_\varepsilon \tilde{S}_t^\delta(\Phi, \Psi) \mathcal{O}_\varepsilon - S_t(\Phi, \Psi) &\subset S_{t+2c_1\ell c^2\varepsilon}((K \cap \mathcal{O}_{\ell c^2\varepsilon})\Phi - \Phi, \Psi(H \cap \mathcal{O}_{\ell c^2\varepsilon})) \\ &\cup (S_{t+2c_1\ell c^2\varepsilon}(\Phi, \Psi) - S_t(\Phi, \Psi)) \\ &\cup S_{t+2c_1\ell c^2\varepsilon}(\Phi, \Psi(H \cap \mathcal{O}_{\ell c^2\varepsilon}) - \Psi). \end{aligned}$$

With respect to the Cartan decomposition $G = KA^+H$, the Haar measure on G is given by $dk \xi(a) da dh$ where dk, da, dh are Haar measures on the components, and ξ is an explicit continuous function. There exists $c_2 > 0$ such that

$$\begin{aligned} &S_{t+2c_1\ell c^2\varepsilon}((K \cap \mathcal{O}_{\ell c^2\varepsilon})\Phi - \Phi, \Psi(H \cap \mathcal{O}_{\ell c^2\varepsilon})) \\ &\subset ((K \cap \mathcal{O}_{\ell c^2\varepsilon})\Phi - \Phi) (S_{t+c_2} \cap A^+) \Psi(H \cap \mathcal{O}_{\ell c^2\varepsilon}). \end{aligned}$$

Hence, it follows that

$$\begin{aligned} & \text{vol}(S_{t+2c_1\ell c^2\varepsilon}((K \cap \mathcal{O}_{\ell c^2\varepsilon})\Phi - \Phi, \Psi(H \cap \mathcal{O}_{\ell c^2\varepsilon}))) \\ & \ll \text{vol}((K \cap \mathcal{O}_{\ell c^2\varepsilon})\Phi - \Phi) \int_{S_{t+c_2} \cap A^+} \xi(a) da \\ & \ll (\ell\varepsilon) \text{vol}(S_t(\Phi, \Psi)) \ll \varepsilon^{1/3} \text{vol}(S_{t+c_3}(\Phi, \Psi)) \end{aligned}$$

for some $c_3 > 0$, where we used (8.1). Similarly,

$$\text{vol}(S_{t+2c_1\ell c^2\varepsilon}(\Phi, \Psi(H \cap \mathcal{O}_{\ell c^2\varepsilon}) - \Psi)) \ll \varepsilon^{1/3} \text{vol}(S_{t+c_3}(\Phi, \Psi)).$$

We will use the Lipschitz property of the function $\phi(t) := \int_{\log \|ka\| \leq t} \xi(a) da$: for sufficiently large t and $\varepsilon \in (0, 1)$,

$$\phi(t + \varepsilon) - \phi(t) \ll \varepsilon \phi(t)$$

uniformly on k . This property can be proved using the argument from [EMS, Appendix] — see [GN, Proposition 7.3]. We obtain

$$\begin{aligned} & \text{vol}(S_{t+2c_1\ell c^2\varepsilon}(\Phi, \Psi) - S_t(\Phi, \Psi)) \\ & \ll \int_{K/M} \left(\int_{t \leq \log \|kav_0\| \leq t+2c_1\ell c^2\varepsilon} \xi(a) da \right) dk \\ & \ll \ell\varepsilon \int_{K/M} \left(\int_{\log \|kav_0\| \leq t} \xi(a) da \right) dk \ll \varepsilon^{1/3} \text{vol}(S_t(\Phi, \Psi)). \end{aligned}$$

Now it follows from (8.4) that

$$\text{vol}(\mathcal{O}_\varepsilon \tilde{S}_t^\delta(\Phi, \Psi) \mathcal{O}_\varepsilon - S_t(\Phi, \Psi)) \ll \varepsilon^{1/3} \text{vol}(S_t(\Phi, \Psi)).$$

Combining this estimate with (8.2), we deduce that

$$\text{vol}(\mathcal{O}_\varepsilon S_t(\Phi, \Psi) \mathcal{O}_\varepsilon - S_t(\Phi, \Psi)) \ll \varepsilon^{1/3} \text{vol}(S_t(\Phi, \Psi)).$$

Similarly, one shows that

$$\text{vol}(S_t(\Phi, \Psi) - \cap_{u,v \in \mathcal{O}_\varepsilon} S_t(\Phi, \Psi)) \ll \varepsilon^{1/3} \text{vol}(S_t(\Phi, \Psi)).$$

Hence, the sets $S_t(\Phi, \Psi)$ are Hölder well-rounded with exponent $1/3$. \square

8.3. Completion of the proofs. We now turn to complete the proofs of the results stated in §8.1.

Proof of Theorem 8.1. By Proposition 8.4, the sets $S_t(\Phi, \Psi)$ are Hölder well-rounded with exponent $1/3$, and by Theorem 4.5, the uniform averages supported on $S_t(\Phi, \Psi)$ satisfy the stable quantitative mean ergodic theorem. Hence, the Theorem 8.1 is a consequence of Theorem 1.9. \square

Proof of Corollary 8.3. Let d be a right-invariant Riemannian metric on $M \backslash H$ and $x_0 \in M \backslash H$ with trivial $(\Gamma \cap H)$ -stabiliser. Define

$$\mathcal{D}_r = \{x \in M \backslash H : d(x, x_0) \leq d(x, x_0\gamma) + r \text{ for } \gamma \in \Gamma \cap H\}.$$

The set \mathcal{D}_0 is the Dirichlet domain for the right $(\Gamma \cap H)$ -action on H . Since $H/(\Gamma \cap H)$ is compact, \mathcal{D}_0 is compact, and it can be defined by finitely many inequalities:

$$\mathcal{D}_0 = \{x \in M \backslash H : d(x, x_0) \leq d(x, x_0\gamma_i) \text{ for } i = 1, \dots, k\}.$$

Note that \mathcal{D}_0 satisfies

$$\mathcal{D}_0\Gamma = M \setminus H \quad \text{and} \quad \text{int}(\mathcal{D}_0)\gamma_1 \cap \text{int}(\mathcal{D}_0)\gamma_2 = \emptyset \quad \text{for } \gamma_1 \neq \gamma_2.$$

We choose a measurable fundamental domain \mathcal{D} for the $(\Gamma \cap H)$ -action on $M \setminus H$ such that $\text{int}(\mathcal{D}_0) \subset \mathcal{D} \subset \mathcal{D}_0$. Note that the map $KA^+/M \rightarrow G/H$ is one-to-one on the set of regular elements. Let Σ be a measurable section of the map $KA^+/M \rightarrow G/H$ which contains all regular elements.

We set

$$T_t(\Phi, \mathcal{D}) = S_t(\Phi, H) \cap \Sigma(\mathcal{D}).$$

We claim that

$$(8.5) \quad |\Gamma H \cap B_t(\Phi)| = |\Gamma \cap T_t(\Phi, \mathcal{D})|.$$

Clearly, for $g \in T_t(\Phi, \mathcal{D})$, we have $gH \in B_t(\Phi)$, and every $x \in \Gamma H \cap B_t(\Phi)$ is of the form γH for some $\gamma \in \Gamma \cap T_t(\Phi, H)$. Moreover, since $H = \mathcal{D}(H \cap \Gamma)$, we can choose $\gamma \in \Gamma \cap T_t(\Phi, \mathcal{D})$. Hence, it remains to show that if $\gamma_1 H = \gamma_2 H$ for some $\gamma_1, \gamma_2 \in \Gamma \cap T_t(\Phi, \mathcal{D})$, then $\gamma_1 = \gamma_2$. We have $\gamma_i = \omega_i h_i \in \Sigma \mathcal{D}$. It follows from the definition of Σ that $\omega_1 = \omega_2$. Hence, $h_1 = h_2(\gamma_2^{-1}\gamma_1)$, and because h_1, h_2 are both in the fundamental domain \mathcal{D} , we conclude that $h_1 = h_2$.

Next, we show that the sets $T_t(\Phi, \mathcal{D})$ are Hölder well-rounded with exponent $1/3$. By Proposition 8.4 (see also Remark 8.5), it remains to check that the set \mathcal{D} is Lipschitz well-rounded, namely, satisfies

$$\text{vol}(\mathcal{D}(\mathcal{O}_\varepsilon \cap H) - \cap_{u \in \mathcal{O}_\varepsilon \cap H} \mathcal{D}u) \ll \varepsilon.$$

For $\gamma \neq e$, the function $f_\gamma(x) = d(x, x_0) - d(x, x_0\gamma)$ is regular on $\{f_\gamma = 0\}$. Hence, for a compact set $\Omega \subset H$,

$$(8.6) \quad \text{vol}(\{x \in \Omega : -\varepsilon < f_\gamma(x) < \varepsilon\}) \ll_\Omega \varepsilon.$$

Also, for $h \in \mathcal{O}_\varepsilon \cap H$,

$$|f_\gamma(xh) - f_\gamma(x)| \ll \varepsilon$$

uniformly on x in compact sets. This implies that for some $c > 0$,

$$\begin{aligned} \mathcal{D}(\mathcal{O}_\varepsilon \cap H) - \cap_{u \in \mathcal{O}_\varepsilon \cap H} \mathcal{D}u &\subset \mathcal{D}_0(\overline{\mathcal{O}_{\varepsilon_1} \cap H}) \cap (\mathcal{D}_{c\varepsilon} - \mathcal{D}_{-c\varepsilon}) \\ &\subset \bigcup_{i=1}^m \{h \in \mathcal{D}_0(\overline{\mathcal{O}_{\varepsilon_1} \cap H}) : -c\varepsilon < f_{\gamma_i}(h) < c\varepsilon\}, \end{aligned}$$

Hence, it follows from (8.6) that the set S is Lipschitz well-rounded. Then by Proposition 8.4, the sets $T_t(\Phi, \mathcal{D})$ are Hölder well-rounded with exponent $1/3$. Now the corollary follows from (8.5) and Theorem 8.1. \square

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