HW2, Bayesian Modelling B 2016/17

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In the homeworks, questions with marks are officially 'exam-style', although you can expect any homework question to appear as an exam question, unless it is explicitly 'not examinable'.

Hand in Q4 and Q5 to be marked. Try the others now or later!

1. Here are some questions about the probability calculus. They test your ability to do mathematics; that is, to present logical and compelling arguments, which combine clarity of expression with elegance and absence of superfluity. Just do a couple of these, to check that you can. I am taking it for granted that you know the propositional calculus (also known as 'zeroth order logic'). If not, you might find the wikipedia page `https://en.wikipedia.org/wiki/Propositional_calculus` helpful.

   Let $A, A_1, A_2, \ldots, B$ be a set of propositions.

   (a) State the three axioms of probability. Distinguish between 'finite' and 'countable' additivity, and show that the latter implies the former.

   (b) Prove each of the following:

      i. $\Pr(\neg A) = 1 - \Pr(A)$.

      ii. If $\{A_1, \ldots, A_n\}$ are mutually exclusive, then

      $$\Pr(A_1 \vee \cdots \vee A_n) = \sum_{i=1}^{n} \Pr(A_i).$$

      iii. If $A \to B$, then $\Pr(A) \leq \Pr(B)$.

      iv. If $\Pr(A) = 0$, then $\Pr(A \wedge B) = 0$.

      v. $\Pr(A \wedge B) \leq \min\{\Pr(A), \Pr(B)\}$. This is known as *Fréchet's inequality.*

2. Let C be a proposition. Prove that if $\Pr(C) > 0$, then $\Pr(\cdot \mid C)$ obeys the three axioms of probability (finite additivity suffices). [10 marks]

3. Let X, Y, Z be collections of random quantities.

   (a) Prove that $X \perp\!\!\!\perp Y \mid Z$ if and only if:

   $$\mathrm{p}(x, y \mid z) = g(x, z) \cdot h(y, z)$$

   for some g, h, whenever $\mathrm{p}(z) > 0$. [10 marks]

(b) Express the statement after the colon in the previous question using predicate logic (i.e., with $\exists$ and $\forall$ quantifiers). [5 marks]

4. Prove that

$$\vDash X \mid Y \iff f_{X|Y}(x \mid y) = \prod_{i=1}^{m} f_{X_i|Y}(x_i \mid y).$$

[10 marks]

5. Let $X = (X_1, \ldots, X_m)$. The PMF f_X is *exchangeable* exactly when

$$\mathrm{p}(x_1, \ldots, x_m) = \mathrm{p}(x_{\pi_1}, \ldots, x_{\pi_m})$$

whenever $(\pi_1, \ldots, \pi_m)$ is a permutation of $(1, \ldots, m)$.

(a) Show that if f_X is exchangeable, then f_{X_A} is exchangeable, where X_A is any subset of X. [10 marks]

(b) Show that if X is conditionally IID given Θ, then X is exchangeable. [10 marks]

(c) Explain why exchangeability can be interpreted as 'similar but not identitical'. [10 marks]