HW3, Bayesian Modelling B 2016/17

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In the homeworks, questions with marks are officially 'exam-style', although you can expect any homework question to appear as an exam question, unless it is explicitly 'not examinable'.

Hand in Q2 and Q4.

1. (a) Consider the prior distribution $\sigma^2 \sim \text{Gamma}(0.001, 0.001)$, where the two parameters are the shape and rate parameters, respectively. Show that under this distribution, $\log \sigma^2$ has an approximately uniform distribution on $\mathbb{R}$. [10 marks]
   (In an exam you would be given the PDF of a Gamma distribution.)

   (b) Show that the following two statements are equivalent:

   i. $\text{p}(x, y) \propto \mathbb{1}(x \in \mathcal{X} \wedge y \in \mathcal{Y})$.

   ii. $X \perp\!\!\!\perp Y$ and X and Y are marginally uniformly distributed.

   [5 marks]

2. Consider the following joint distribution for $X = (X_1, \ldots, X_5)$:

$$p(x) = p(x_1) \cdot \text{p}(x_2 \mid x_1) \cdot \text{p}(x_3) \cdot \text{p}(x_4 \mid x_1, x_3) \cdot \text{p}(x_5 \mid x_3).$$

   (a) Draw the DAG of f_X.

   (b) Draw the CIG of f_X.

   (c) Answer True or False to the following statements:

   i. $X_2 \perp\!\!\!\perp X_3, X_4 \mid X_1$

   ii. $X_1 \perp\!\!\!\perp X_2 \mid X_3, X_4$

   iii. $X_5 \perp\!\!\!\perp X_1, X_4 \mid X_3$

   iv. $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3, X_5$

   For each statement, state whether or not you could provide an answer directly from the DAG (i.e. without constructing the CIG).

   (d) Draw the DAG of (X_2, X_3, X_4, X_5), i.e. after marginalizing over X_1.

3. (a) Let $X = (X_1, \ldots, X_m)$. Give definitions for the DAG of f_X and the CIG of f_X. State and prove the Moralization Theorem. [15 marks]

   (b) State the Hammersley-Clifford Theorem, and explain its role in interpeting the CIG. [10 marks]

4. Consider the 'old-fashioned' regression model

$$Y_i = \alpha + \beta\, X_i + \epsilon_i, \qquad i = 1, \ldots, n,$$

where $\epsilon \overset{\text{iid}}{\sim} \text{Normal}(0, \sigma^2)$.

(a) Write this model as a DAG, using a plate. Hint: α, β, and σ^2 are parameters but ϵ is not. [5 marks]

(b) Generalize this DAG so that each case gets its own (α_i, β_i), where the α's and β's are each exchangeable. [5 marks]

(c) Make explicit choices for the marginal and conditional distributions in the DAG, and identify the restriction that forces the generalized model to behave like the old-fashioned one. [5 marks]