Quantifying uncertainty in Probability of Exceedence (PE) curves

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Climate, weather, and flooding

Climate is the distribution of weather, and a natural hazard like a flood is an extreme weather event.

- *A priori:* The distribution of 'weather events' (e.g. storms) over the coming year is uncertain; hence we imagine a collection of possible storms, Ω, and a matching collection of probabilities, $P := \{p_\omega : \omega \in \Omega\}$.

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- *A posteriori:* For a given storm ω, we can compute a single scalar summary, v_ω:

$$\overbrace{\omega}^{\text{storm}} \longrightarrow \overbrace{h_\omega(\mathbf{x}, t)}^{\text{footprint}} \longrightarrow \overbrace{\max_t \big|h_\omega(\mathbf{x}_0, t)\big|}^{\text{summary}} =: v_\omega$$

where v_ω might be the maximum depth of water upstream of a bridge in a town centre (location $\mathbf{x}_0$).

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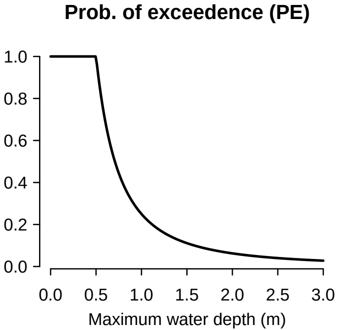

The probability distribution function of $\tilde{v}$ is

$$F_{\tilde{v}}(v) := \Pr\{\tilde{v} \leq v\} = \sum_{\omega} 1[v_\omega \leq v] p_\omega .$$

The PE curve is the plot of v (x-axis) against $1 - F_{\tilde{v}}(v)$.

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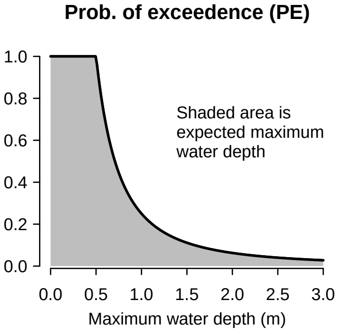

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Uncertainty from sampling

Often, we cannot evaluate the probability distribution function of $\tilde{v}$ exactly, and have to approximate by sampling.

In other words, we generate a large number N of independent realizations of $\tilde{v}$, and use the resulting empirical distribution function as an estimate of the true distribution function:

$$\tilde{v}^{(1)}, \ldots, \tilde{v}^{(N)} \overset{\text{iid}}{\sim} F_{\tilde{v}} \quad \text{and} \quad \hat{F}_{\tilde{v}}(v) = N^{-1} \sum_{i=1}^{N} 1[\tilde{v}^{(i)} \leq v]$$

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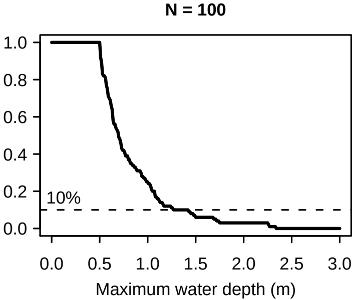
- Because this is only an estimate, we should quantify our uncertainty about the true PE curve in terms of $1-\alpha$ confidence bands, such that

  $$\Pr\left\{L(x) \leq F_{\tilde{v}}(x) \leq U(x) \text{ for all } x\right\} \geq 1-\alpha.$$

  The width of $U(x) - L(x)$ will depend on N and α. The functions L and U can be specified using the Dvoretzky- Kiefer-Wolfowitz inequality.

PE curve uncertainty from sampling, $\alpha = 5\%$



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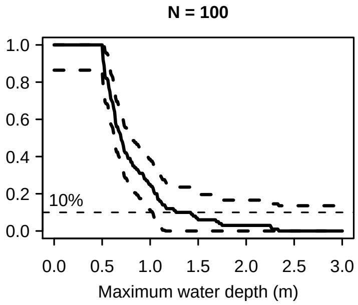

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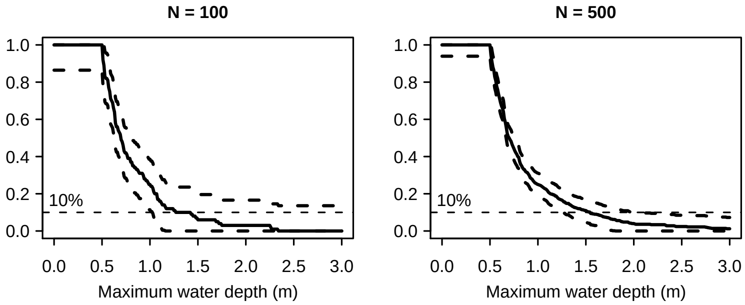

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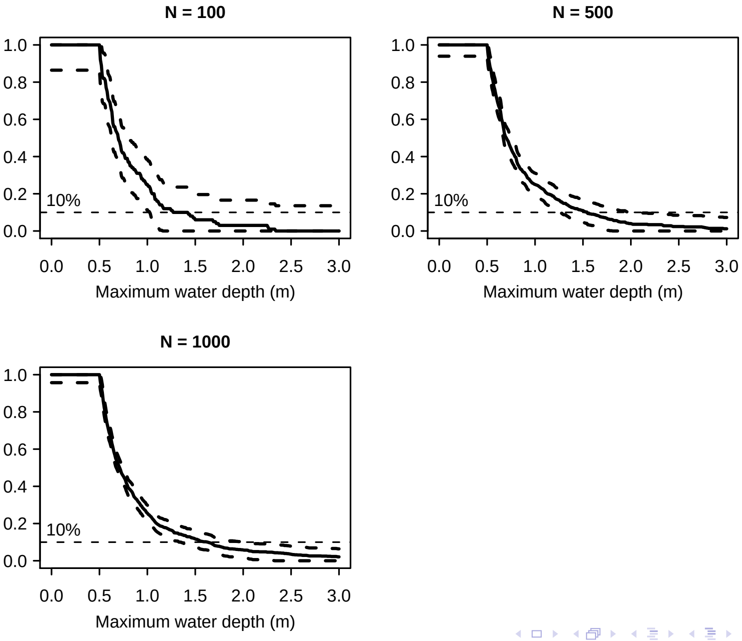

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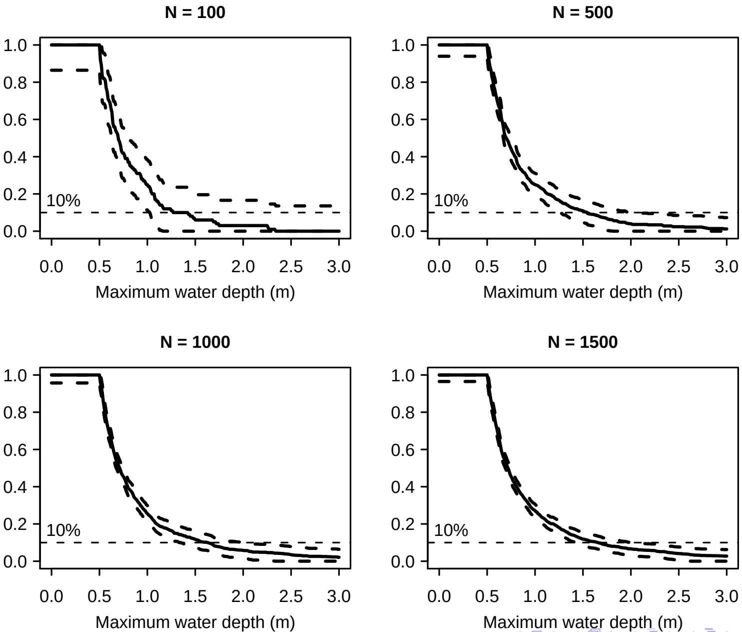

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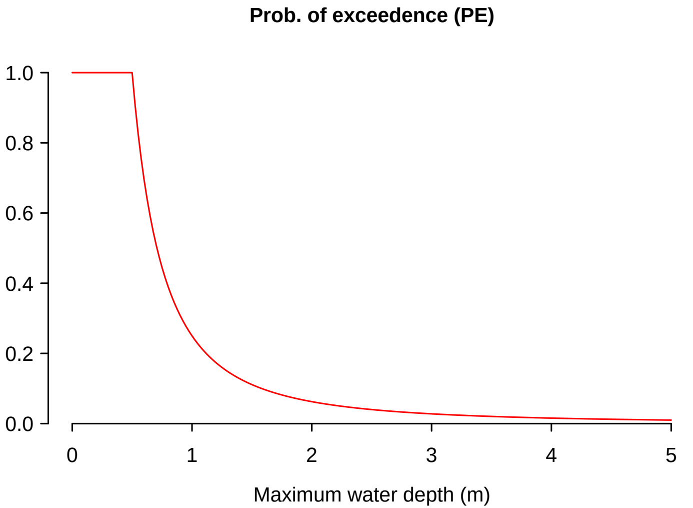

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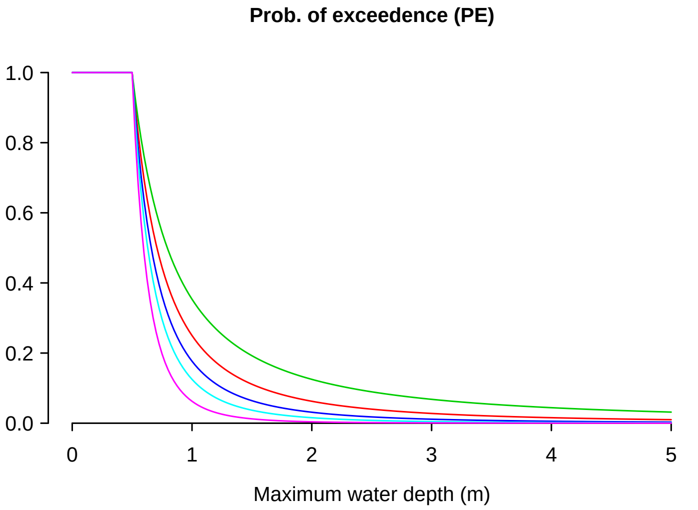

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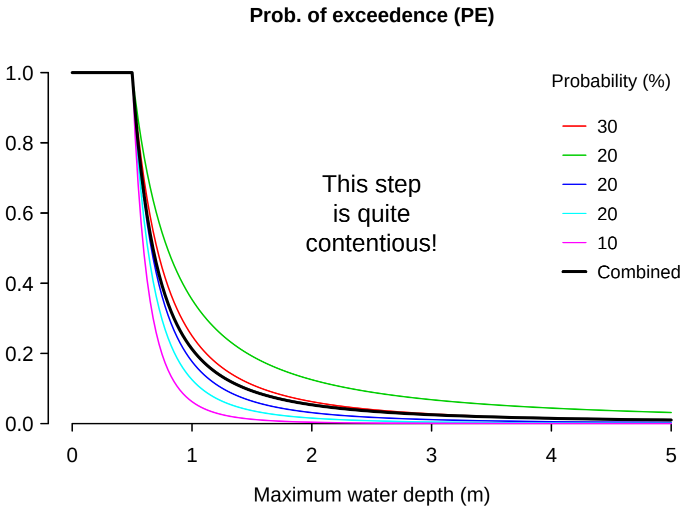

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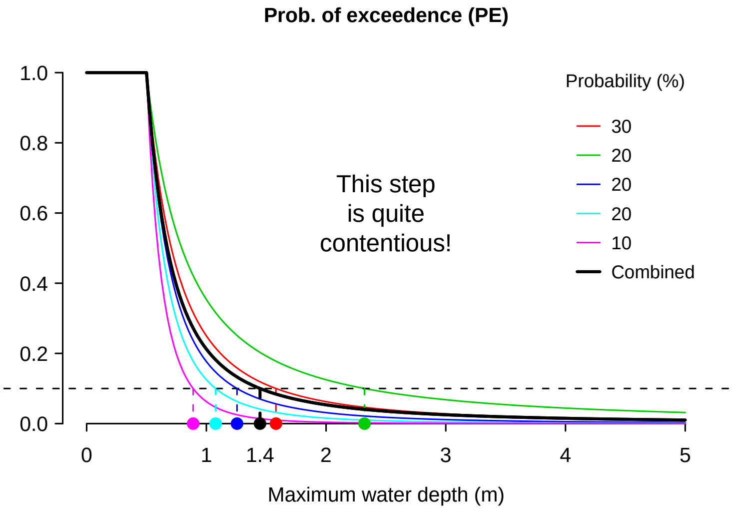

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2. Uncertainty about the hazard process, e.g. uncertainty about future climate.
3. Uncertainty about the physical processes, e.g. the footprint function. *Is this the largest uncertainty of all?*