Quantifying uncertainty in Probability of Exceedence (PE) curves

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Climate, weather, and flooding

Climate is the distribution of weather, and a natural hazard like a flood is an extreme weather event.

- A priori: The distribution of ‘weather events’ (e.g. storms) over the coming year is uncertain; hence we imagine a collection of possible storms, $\Omega$, and a matching collection of probabilities, $P := \{p_\omega : \omega \in \Omega\}$.

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\[
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\]

- **A posteriori:** For a given storm \( \omega \), we can compute a single scalar summary, \( v_\omega \):

\[
\begin{align*}
\omega & \quad \longrightarrow \quad h_\omega(x, t) \quad \longrightarrow \quad \max_t |h_\omega(x_0, t)| =: v_\omega \\
\text{storm} & \quad \text{footprint} & \quad \text{summary}
\end{align*}
\]

where \( v_\omega \) might be the maximum depth of water upstream of a bridge in a town centre (location \( x_0 \)).
Probability of exceedence (PE) curves

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The probability distribution function of $\tilde{\nu}$ is

$$F_{\tilde{\nu}}(v) := \Pr\{\tilde{\nu} \leq v\} = \sum_{\omega} 1[\nu_{\omega} \leq v]p_{\omega}.$$

The PE curve is the plot of $v$ (x-axis) against $1 - F_{\tilde{\nu}}(v)$.
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Uncertainty from sampling

Often, we cannot evaluate the probability distribution function of $\tilde{\nu}$ exactly, and have to approximate by sampling.

In other words, we generate a large number $N$ of independent realizations of $\tilde{\nu}$, and use the resulting empirical distribution function as an estimate of the true distribution function:

$$\tilde{\nu}(1), \ldots, \tilde{\nu}(N) \overset{iid}{\sim} F_{\tilde{\nu}} \quad \text{and} \quad \hat{F}_{\tilde{\nu}}(\nu) = N^{-1} \sum_{i=1}^{N} 1[\tilde{\nu}(i) \leq \nu]$$
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\]

Because this is only an estimate, we should quantify our uncertainty about the true PE curve in terms of \( 1 - \alpha \) confidence bands, such that

\[
\Pr \left\{ L(x) \leq F_{\tilde{v}}(x) \leq U(x) \text{ for all } x \right\} \geq 1 - \alpha.
\]

The width of \( U(x) - L(x) \) will depend on \( N \) and \( \alpha \). The functions \( L \) and \( U \) can be specified using the Dvoretzky- Kiefer-Wolfowitz inequality.
PE curve uncertainty from sampling, $\alpha = 5\%$
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$N = 100$

Maximum water depth (m)

10%

0.0 0.5 1.0 1.5 2.0 2.5 3.0

0.0 0.2 0.4 0.6 0.8 1.0
PE curve uncertainty from sampling, $\alpha = 5\%$

![Graph of PE curve uncertainty from sampling for N = 100 and N = 500. The graphs show the relationship between maximum water depth (m) and the probability of exceedance (PE). The uncertainty is indicated by the range of lines for different sample sizes.](image-url)
PE curve uncertainty from sampling, $\alpha = 5\%$

N = 100

N = 500

N = 1000

Maximum water depth (m)
PE curve uncertainty from sampling, $\alpha = 5\%$

- **N = 100**
- **N = 500**
- **N = 1000**
- **N = 1500**
Uncertainty about climate

Uncertainty about climate means multiple candidates for climate, which means multiple PE curves.
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![Graph showing the probability of exceedance (PE) versus maximum water depth (m). The graph is a curve that decreases sharply as the maximum water depth increases.](image-url)
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2. Uncertainty about the hazard process, e.g. uncertainty about future climate.
3. Uncertainty about the physical processes, e.g. the footprint function. Is this the largest uncertainty of all?