Complex systems: Accounting for model limitations

Jonathan Rougier

Department of Mathematics
University of Bristol, UK

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Illustration: the Greenland ice-sheet

- Local climate
- Greenland
- Local seas
- Global climate
- Global oceans
- Whales!
Simplest interesting example

Conditional on $\theta$:

$$x_0 \sim \pi_{x_0}(\cdot ; \theta)$$  \hspace{1cm} \text{(init. cond. unc.)}
$$x_t = g(x_{t-1}, \omega_t ; \theta)$$  \hspace{1cm} \text{(state eqn.)}
$$y_t = f(x_t ; \theta) + \nu_t$$  \hspace{1cm} \text{(obs. eqn.)}

where

$$\omega_t \overset{iid}{\sim} \mathcal{N}(0, I)$$  \hspace{1cm} \text{(structural uncertainty)}
$$\nu_t \overset{iid}{\sim} \mathcal{N}(0, \nu^2)$$  \hspace{1cm} \text{(measurement unc.)}

and then let $\theta \sim \pi_{\theta}(\cdot)$, to account for parametric uncertainty. The functions $f$ and $g$ are given, likewise the measurement uncertainty standard deviation, $\nu$.

Sampling from $\{x_0, x_1, \ldots, x_T, \theta\} \mid \{y_1, \ldots, y_T\}$ “intractable and unsolved” (C. Andrieu)
The calibration problem

To learn about $\theta$, typically by summarising samples from the distribution $\pi(\theta \mid y)$, where $y = (y_1, \ldots, y_T)$. We’ll treat $x_0$ as known, for simplicity.

- Ideally, we would run an MCMC chain with proposal $q(\theta \rightarrow \theta')$ and acceptance probability

$$\alpha(\theta, \theta') = 1 \land \frac{\pi(y \mid \theta') \pi(\theta')}{\pi(y \mid \theta) \pi(\theta)} \frac{q(\theta' \rightarrow \theta)}{q(\theta \rightarrow \theta')}.$$
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- The catch is that we need to integrate out $\omega$ in order to evaluate $\pi(y | \theta)$:

$$\pi(y | \theta) = \int \pi(y | \omega, \theta) \pi(\omega) \, d\omega$$

where $\omega = (\omega_1, \ldots, \omega_T) — ouch!!$
The calibration problem (cont.)

In a picture . . .

$(\omega \mid \theta)$-space (high-dimensional)

$\theta$-space (low-dimensional)
We could approximate this tricky density with an Importance Sampler estimate

\[ \tilde{\pi}(y \mid \theta) = N^{-1} \sum_{i=1}^{N} \frac{\pi(y \mid \omega^i, \theta) \pi(\omega^i \mid \theta)}{q_\omega(\omega^i; y, \theta)} \quad \omega^i \overset{iid}{\sim} q_\omega(\cdot; y, \theta), \]

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Mark Beaumont’s (2003) result. The \( \theta \)-marginal of the MCMC equilibrium distribution with \( \pi(y \mid \theta) \) replaced by \( \tilde{\pi}(y \mid \theta) \) is still \( \pi(\theta \mid y) \), for all \( N \geq 1 \).

1. One has to accept/reject \( \{\omega^i\} \) along with \( \theta \).
2. Small \( N \) normally implies a sticky chain.

The general result was stated by Andrieu et al (2007): the \( \theta \)-marginal is \( \pi(\theta \mid y) \) for any unbiased estimator of \( \pi(y \mid \theta) \).
The stochastic van der Pol oscillator

Has a ‘slow’ response $x$ and a ‘fast’ response $x'$, related as

$$x'' + x' + (\alpha - x^2)x = \sigma x \, dW,$$

where $W$ is a Brownian motion. It is the basis for several phenomenological models of glacial cycles, where $x$ is ‘ice volume’ and $x'$ is ‘temperature’ (or, effectively, ‘CO$_2$’). Here $\theta = (\alpha, \sigma)$. 
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One realisation:
The evidence (24 hours)
Summary

- In inference for environmental systems, model limitations require us to account for both parametric and structural uncertainty.

- The generic problem for dynamical systems is therefore non-linear data assimilation with uncertain static parameters.

- Learning about the parameters involves integrating out the high-dimensional state vector; this can be done ‘exactly’ (in the MCMC sense) using Beaumont’s result.

- Recent developments not mentioned here have generalised this approach (e.g. pseudo-marginal approach, particle-MCMC).

- This is an exciting time to be working as a statistician in environmental science!