HW1, Theory of Inference 2015/6

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A general statement on homeworks. These homeworks are an opportunity for you to develop your understanding, and to practice your maths and communication skills. If you hand-in your homeworks, you will get feedback on how well you are doing. You are *strongly encouraged* to hand-in your homeworks.

It is crucial that you express your answers clearly, in well-structured sentences. Where you are writing maths, your writing must be tidy enough that there can be no ambiguity about symbols and the names of variables. The way you lay-out your maths must be logical and clear, using indentation, alignment, and other standard conventions. This is a skill you must master before you leave the University. Future employers and colleagues will rightly be critical of sloppy thinking and sloppy communicating.

I am told by students that my marking criteria are very strict. Please be absolutely clear that I am marking according to the the criteria that you will be judged by when you leave the University. Do not be put off by low marks. Come to an Office Hour to discuss how you could have done better, and study the solutions.

In the following questions, I show marks in square brackets, to give you an idea of the approximate tariff the question would carry in an exam.

1. Study the Dutch book argument for conditional probabilities (sec. 1.8 in the handout). Then adapt the proof to show that, under coherence

   (a) If Q implies P, then $0 < p = q < r = 1$,

Answer. If Q implies P, then $\neg P \wedge Q$ is impossible, and the third row from M is dropped, to give

$$M := \begin{array}{r} \\ \neg P, \neg Q \\ P, \neg Q \\ P,\ \ Q \end{array} \begin{array}{c} \begin{array}{ccc} P \wedge Q & Q & P \mid Q \end{array} \\ \begin{pmatrix} -p & -q & 0 \\ -p & -q & 0 \\ 1-p & 1-q & 1-r \end{pmatrix} \end{array}.$$

Multiplying out $M^T \boldsymbol{x} = \mathbf{0}$ then gives

$$\begin{aligned} x_3 - p \cdot s &= 0 \\ x_3 - q \cdot s &= 0 \\ x_3 \cdot (1-r) &= 0, \end{aligned}$$

where $s := x_1 + x_2 + x_3$. If $\boldsymbol{x} \gg \mathbf{0}$ (for coherence), then $s > 0$ and $0 < p = q < 1$, $r = 1$, as required.

(b) If Q implies $\neg P$, then $0 = p = r < q < 1$.

Answer. If Q implies $\neg P$, then $P \wedge Q$ is impossible, and the fourth row of M is dropped, to give

$$M := \begin{array}{r} \\ \neg P, \neg Q \\ P, \neg Q \\ \neg P,\ \ Q \end{array} \begin{array}{c} \begin{array}{ccc} P \wedge Q & Q & P \mid Q \end{array} \\ \begin{pmatrix} -p & -q & 0 \\ -p & -q & 0 \\ -p & 1-q & -r. \end{pmatrix} \end{array}$$

Multiplying out $M^T \boldsymbol{x} = \mathbf{0}$ then gives

$$\begin{aligned} -p \cdot s &= 0 \\ x_3 - q \cdot s &= 0 \\ -x_3 \cdot r &= 0, \end{aligned}$$

where $s := x_1 + x_2 + x_3$. Hence $\boldsymbol{x} \gg \mathbf{0}$ implies that $p = r = 0$, and $0 < q < 1$, as required.

Insert these values into eq. (1.39) in the handout to check them for sense. [10 marks]

Answer. In the first case, where Q implies P, we have

$$\Pr(P, Q) = \Pr(P \mid Q) \cdot \Pr(Q) \quad \text{or} \quad p = 1 \cdot p.$$

In the second case, where Q implies $\neg P$, we have

$$\Pr(P, Q) = \Pr(P \mid Q) \cdot \Pr(Q) \quad \text{or} \quad 0 = 0 \cdot q.$$

So both of these make sense. We observe that conditional probabilities preserve logical relationships; i.e. if Q implies P then $\Pr(P \mid Q) = 1$, and if Q implies $\neg P$ then $\Pr(P \mid Q) = 1 - \Pr(\neg P \mid Q) = 1 - 1 = 0$.

2. In the notes, my statement of the converse case in the Dutch book argument for conditional probabilities is messy. Try to do better. There is a piece of cake for successful attempts.

   Answer. Have a look at the updated handout for a clearer presentation of the proofs, for which the logic is clear, although it is also a bit complicated.

3. The Mayor of Bristol would like to know about the probability of damage to the nuclear power station at Hinkley Point in the event of a Magnitude 4 ashy eruption in Iceland, into a north-westerly wind. After working with a team of volcanologists and meteorologists, you report that your team's probability is 0.0015.[1] Describe the propositions on which this value is based. Give an informal account to the Mayor of what this number represents. [10 marks]

   Answer. This is a question about a conditional probability, with "in the event of a Magnitude 4 ashy eruption in Iceland, into a north-westerly wind" being a hypothetical. So this would be E in probability statements of the form $\Pr(\cdot \mid E)$. "Damage" is not specified. There are many different ways

[1] I made this number up!

in which a nuclear power station such as Hinkley Point can be damaged. Denote these by $D_1, \ldots, D_k$. "Damage" represents the disjunction of these, i.e. $D = \bigvee_{i=1}^{k} D_i$. So the statement has the propositional form $\Pr(D \mid E) = 0.0015$.

Frankly, this is such a hard question that you can get 10 marks for just showing up. The following comments are my reflections and are in no way official statistics—no one has a good answer to his question, but that should not stop us from trying.

First, there is a subtle issue of interpretation. Implicit in the question, but nonetheless ambiguous, is that the damage is caused by the eruption. If we fix a time-frame, such as week (e.g. roughly the duration of the eruption), there is a base-rate for damage, which gives $\Pr(D)$. We can assume that $\Pr(D \mid E) > \Pr(D)$. What is not clear is whether the Mayor wants to know $\Pr(D|E)$ or $\Pr(D|E) - \Pr(D)$. The latter would be, approximately, the probability of damage caused by the eruption, were it to happen (this is known as 'attributable risk' in Epidemiology). But let's assume the Mayor and the operator of the power station want $\Pr(D \mid E)$. This would be more usual, on the basis that the power station would be turned off if the probability of damage surpassed some threshold. But it includes the base rate.

It is very difficult to explain conditional probabilities. Bets are something that most people can manage. Called-off bets are much more complicated. I think, in this case, it is better to use hypothetical worlds. One can prove that the two intepretations are the same, once we make it clear what we mean by 'hypothetical world', and how we should construct beliefs about hypothetical worlds. I am *not* covering this in the lectures! So I might say something like this: "Imagine that the eruption had happened, but that everything else was as consistent as possible with today. In that case I would only pay 0.15p for a bet to win £1 if there was damage, and nothing otherwise." Here I am locating the thought experiment in the 'nearest possible world' to today, except that the eruption has happened.

The value 0.15p is small and hard to grasp; it might be confused with 15p, which is 100 times larger. It is helpful to relate it to a more tangible gamble, such as rolling a dice, tossing a coin, or spinning a roulette wheel. 0.0015 is about 1/700, which is quite a hard number to get to. But it is between

1/1024 and 1/512, so I would continue "This is more than I would pay to win £1 on ten heads in ten tosses, but less than I would pay to win £1 on nine heads in nine tosses. That is, I think that damage is a little less probable than nine heads in nine tosses."

There are no 'right' answers to this question. It simply serves to illustrate that it is *very* hard to communicate effectively about uncertainty, especially for improbable events.