HW3, Theory of Inference 2015/6

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Here is a homework about Decision Theory, concerning a real and important decision. You are *strongly encouraged* to do this homework.

The London VAAC (`http://www.metoffice.gov.uk/aviation/vaac/`) assesses the ash concentration in the eastern N. Atlantic. We will consider one pixel, say the one at $(54, -31)$–$(55, -30)$. This pixel is either ashy ($X = \mathsf{true}$) or not-ashy ($X = \mathsf{false}$). A satellite retrieval provides an imperfect test of whether the pixel is ashy. This test is either positive ($Y = +$) or negative ($Y = -$), with + favouring the presence of ash.

1. Define the 'sensitivity' and the 'specificity' of the retrieval; denote these as α and β in what follows. Propose a reason why the sensitivity might be high, and the specificity low. [5 marks]

2. What additional information do we need in order to compute the posterior probability of $X = \mathsf{true}$ given a positive test? (Clue: denote this quantity as π_+.) Give the formulae for the posterior probabilities of X conditional on $Y = +$. (You may want to express these as odds.) [10 marks]

3. Let the action set be $\mathcal{A} := \{\mathsf{safe}, \mathsf{unsafe}\}$. Denote the loss function as

$L(a, x)$	$x = \mathsf{false}$	$x = \mathsf{true}$
$a = \mathsf{safe}$	ℓ_{00}	ℓ_{01}
$a = \mathsf{unsafe}$	ℓ_{10}	ℓ_{11}

What are the natural constraints on the values of the four ℓ's? [5 marks]

4. State the Bayes Rule Theorem. Prove that the Bayes Rule for this problem with $y = +$ is

$$\delta^*(+) = \begin{cases} \mathsf{safe} & O(+) < \frac{\ell_{10}-\ell_{00}}{\ell_{01}-\ell_{11}} \\ \mathsf{unsafe} & \text{otherwise,} \end{cases} \tag{1}$$

where

$$O(+) := \frac{\Pr(X = \mathsf{true} \mid Y = +)}{\Pr(X = \mathsf{false} \mid Y = +)},$$

termed the *posterior odds* for X when $Y = +$. [15 marks]

5. Interpret this result. [5 marks]

6. The test shows $Y = +$. Using the values $\alpha = 0.95$, $\beta = 0.3$, $\pi_+ = 0.05$, $\ell_{00} = \ell_{11} = 0$, $\ell_{01}/\ell_{10} = 10$, what is the optimal action? Comment on this result. [5 marks]