HW2, Theory of Inference 2016/7

Jonathan Rougier
School of Mathematics
University of Bristol UK

In the following questions, I show marks in square brackets, to give you an idea of the approximate tariff the question would carry in an exam.

Here are some questions about certification of algorithms for point-estimation. Note below that certification is always for an algorithm *and* a model. You do not need to hand in Q2d.

1. Consider the model in which $X_1, \dots, X_m$ are IID, which we express as

$$\mathcal{E}^m = \{\mathcal{X}^m, \Omega, f\}$$

where $f(x; \theta) = \prod_{i=1}^m f_1(x_i; \theta)$ for some f_1.

(a) Show that that if $m(\theta)$ is the expectation of X_1 (i.e. of each X_i), then

$$\tilde{\theta}_\pi(x) = \frac{1}{n} \sum_{j=1}^n x_{\pi_j} \tag{1}$$

is an unbiased estimator of $m(\theta)$ in $\mathcal{E}^m$, where $\pi = (\pi_1, \dots, \pi_n)$ is any size-n subset of $\{1, \dots, m\}$. Hint: take $\pi = (1, \dots, n)$, without loss of generality. [10 marks]

(b) Show that, for any fixed π,

$$s_\pi^2(x) = \frac{1}{n-1} \sum_{j=1}^n \left\{ x_{\pi_j} - \tilde{\theta}_\pi(x) \right\}^2$$

is an unbiased estimator of the variance of X_1 for $\mathcal{E}^m$. [10 marks]

(c) Show that $\sqrt{s_\pi^2}$ is *not* an unbiased estimator of the standard deviation of X_1 for $\mathcal{E}^m$. Hint: Jensen's inequality. [10 marks]

2. Consider the model $\mathcal{E} = \{\mathcal{X}, \Omega, f\}$. Point estimators of scalar functions of θ are often certified by their Mean Squared Error (MSE) function. If $\hat{g}$ is a point estimator of $g(\theta) \in \mathbb{R}$, then

$$\mathrm{MSE}_{\hat{g}}(\theta) := \mathrm{E}[\{\hat{g}(X) - g(\theta)\}^2; \theta]$$

is the MSE of $\hat{g}$.

(a) Express $\mathrm{MSE}_{\hat{g}}$ as an explicit equation involving the components of $\mathcal{E}$, and $\hat{g}$ and θ. [5 marks]

(b) Show that

$$\mathrm{MSE}_{\hat{g}}(\theta) = \mathrm{bias}_{\hat{g}}(\theta)^2 + \mathrm{Var}_{\hat{g}}(\theta)$$

giving exact definitions for the two functions on the righthand side. Hint: consider introducing the quantity $\bar{g}(\theta) = \mathrm{E}\{\hat{g}(X); \theta\}$. [10 marks]

(c) Compute the MSE of the estimator $\tilde{\theta}$ in (1). [10 marks]

(d) (Not examinable.) Let $\mathcal{E}^m$ be the IID model. An estimator $\hat{g}_m$ of $g(\theta)$ is certified as *consistent* for $\mathcal{E}^m$ exactly when

$$\forall \theta \in \Omega, \forall \epsilon > 0 \quad \lim_{m \to \infty} \Pr\left\{|\hat{g}_m(X_{1:m}) - g(\theta)| \geq \epsilon; \theta\right\} = 0.$$

Show that $\hat{g}_m$ is consistent if and only if, for all $\theta \in \Omega$, $\hat{g}_m$ converges to a point in Ω and its bias converges to zero.