MT&I: Exercises 6

1. Let ν be a charge on $(X, \mathbb{X})$.

   a) Let $\mathbb{X} \ni E_n \uparrow E$ (this means for all n, $E_n \subset E_{n+1}$). Show that $\nu(E) = \lim \nu(E_n)$.

   b) Let $\mathbb{X} \ni F_n \downarrow F$ (this means for all n, $F_n \supset F_{n+1}$). Show that $\nu(F) = \lim \nu(F_n)$.

2. Let ν be a charge on $(X, \mathbb{X})$. Show that

   a) $\nu^+(E) = \sup\{\nu(F) : \mathbb{X} \ni F \subseteq E\}$,

   b) $\nu^-(E) = -\inf\{\nu(F) : \mathbb{X} \ni F \subseteq E\}$.

3. Let $(X, \mathbb{X}, \mu)$ be a measure space and $f \in L(X, \mathbb{X}, \mu)$. Let $\nu : \mathbb{X} \to \mathbb{R}$ be the charge given by $\nu(A) = \int_A f \mathrm{d}\mu$. Show that a set E is null with respect to ν if and only if $\mu(E \cap \{x \in X : f(x) \neq 0\}) = 0$.

4. Prove Theorem 8.7 from the lecture notes. That is if $f \in L(X, \mathbb{X}, \mu)$ and $\nu : \mathbb{X} \to \mathbb{R}$ is the charge given by $\nu(A) = \int_A f \mathrm{d}\mu$ then show that the positive and negative variations of ν, are given by $\nu^+(A) = \int_A f^+ \mathrm{d}\mu$ and $\nu^- = \int_A f^- \mathrm{d}\mu$ respectively.

5. Let $\nu(E) = \int_E xe^{-x^2} \mathrm{d}\lambda$ ($E \in \mathbb{B}$, λ is Lebesgue measure). Give a Hahn decomposition of $\mathbb{R}$ with respect to ν.

6. Let ν, μ be σ-finite measures on $(X, \mathbb{X})$ with $\nu << \mu$. Let $f = \frac{d\nu}{d\mu} \in M^+$. Show that for any $g \in M^+$,

   $$\int g \, d\nu = \int gf \, d\mu.$$

   Hint: Apply Monotone Convergence Theorem to simple functions.

7. Let ν, λ, μ be σ-finite measures on $(X, \mathbb{X})$ with $\nu << \lambda$ and $\lambda << \mu$. Show that $\nu << \mu$ and

   $$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu} \qquad \mu\text{-a.e.}$$

8. Let

   $$f(x) = \begin{cases} \sqrt{1-x}, & x \leq 1 \\ 0, & x > 1 \end{cases}$$

   and

   $$g(x) = \begin{cases} x^2, & x \leq 0 \\ 0, & x > 0 \end{cases}$$

   Let

   $$\nu(E) = \int_E f \, d\lambda \quad \text{and} \quad \mu(E) = \int_E g \, d\lambda \quad (E \in \mathbb{B})$$

   Find the Lebesgue decomposition of ν with respect to μ.

9. Let $(X, \mathbb{X}_0, \mu)$ be a probability space, $f : X \to \mathbb{R}$ an integrable function and $\mathbb{X} \subset \mathbb{X}_0$ a sigma algebra. Show that there exists a function g which is integrable with respect to the measurable space $(X, \mathbb{X})$ and for which any $A \in \mathbb{X}$ satisfies $\int_A f \mathrm{d}\mu = \int_A g \mathrm{d}\mu$. (This function (random variable) is known as the conditional expectation).