Analysis 3: HW4

1.

i) Let $\alpha > 1$, $t \geq 0$. Prove that $0 \leq \frac{t}{1+t^\alpha} \leq 1$ and deduce that

$$\lim_{n\to\infty} \int_{[0,2\pi]} \frac{nx \sin x}{1+(nx)^\alpha}\, dm = 0.$$

ii) Let $n \geq 2$, $x \geq 0$. Prove that $(1 + x/n)^n \geq (1/4)x^2$. and deduce that $x \to (1 + x/n)^{-n}$ is Lebesgue-integrable over $[1, \infty)$ and that

$$\lim_{n\to\infty} \int_{[1,\infty)} (1 + \frac{x}{n})^{-n}\, dm = e^{-1}.$$

2. Let $C[0, 1]$ be the linear space of continuous functions on $[0, 1]$ to $\mathbb{R}$ and define N_1 to be the Riemann integral of $|f|$ over $[0, 1]$. Show that N_1 is a norm on $C[0, 1]$. If f_n is defined by

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq (1 - 1/n)/2, \\ \text{linear} & \text{for } (1 - 1/n)/2 \leq x \leq 1/2; \\ 1 & \text{for } 1/2 \leq x \leq 1, \end{cases}$$

show that (f_n) is a Cauchy sequence, but that it does not converge to an element of $C[0, 1]$.

3. Let N be a norm on a linear space V and d be defined for $u, v \in V$ by $d(u, v) = N(u - v)$. Show that d is a metric on V; that is, for all $u, v, w \in V$,

i) $d(u, v) \geq 0$;

ii) $d(u, v) = 0$ if and only if $u = v$;

iii) $d(u, v) = d(v, u)$;

iv) $d(u, v) \leq d(u, w) + d(w, v)$.

4. If $f \in L^1$ and $\epsilon > 0$ then there exists a simple $\mathfrak{A}$-measurable function φ such that $\|f - \varphi\|_1 < \epsilon$. Extend this to L^p, $1 \leq p < \infty$. Is this true for L^∞?

5. Let $\Omega = \mathbb{N}$ and λ a measure on $\mathfrak{A} = \mathcal{P}(\Omega)$ defined by

$$\lambda(E) = \sum_{n\in E} (1/n^2),\ E \in \mathfrak{A}.$$

(a) Show that $f : \Omega \to \mathbb{R}$, $f(n) = \sqrt{n}$ satisfies $f \in L^p$ if and only if $1 \leq p < 2$.

(b) Find a function f such that $f \in L^p$ if and only if $1 \leq p \leq p_0$.

6. Let $(\Omega, \mathfrak{A}, \mu)$ be a finite measure space. If f is $\mathfrak{A}$-measurable, let $E_n = \{(n - 1) \leq |f| < n\}$. Show that $f \in L^1$ if and only if

$$\sum_{n=1}^{\infty} n\mu(E_n) < +\infty.$$

More generally, $f \in L^p$ for $1 \leq p < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^p \mu(E_n) < +\infty.$$

7. If $(\Omega, \mathfrak{A}, \mu)$ is a finite measure space and $f \in L^p$ then $f \in L^r$ for any $1 \le r < p$ and

$$\|f\|_r \le \mu(\Omega)^{\frac{1}{r}-\frac{1}{p}} \|f\|_p.$$

8. Suppose that $\Omega = \mathbb{N}$ and μ is the counting measure on Ω. If $f \in L^p$ then $f \in L^s$ with $1 \le p \le s < \infty$ and $\|f\|_s \le \|f\|_p$.

9. Let $(\Omega, \mathfrak{A}, \mu) = (\mathbb{R}, \mathfrak{B}, m)$ and $f(x) = |x|^{-1/2}(1 + |\log x|)^{-1}$. Then $f \in L^p$ if and only if $p = 2$.

10. Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space and suppose $f \in L^{p_1}$ and $f \in L^{p_2}$ with $1 \le p_1 < p_2 < \infty$. Prove that $f \in L^p$ for any p with $p_1 \le p \le p_2$.

11. Let $p > 1$, $f \ge 0$, $f \in L^p((0, \infty))$ and $F(x) = \int_{[0,x]} f \, dm$. Show that if p and q are conjugate indices, then $F(x) = o(x^{1/q})$ as $x \to 0$ and as $x \to \infty$.

12. If $f \in L^\infty(\Omega, \mathfrak{A}, \mu)$ then $|f(\omega)| \le \|f\|_\infty$ for almost all ω. Moreover, if $A < \|f\|_\infty$ then there exists a set $E \in \mathfrak{A}$ with $\mu(E) > 0$ such that $|f(\omega)| > A$ for all $\omega \in E$.

13. If $f \in L^p$, $1 \le p \le \infty$ and $g \in L^\infty$ then the product $fg \in L^p$ and $\|fg\|_p \le \|f\|_p \|g\|_\infty$.

14. The space $L^\infty(\Omega, \mathfrak{A}, \mu)$ is contained in $L^1(\Omega, \mathfrak{A}, \mu)$ if and only if $\mu(\Omega) < \infty$. If $\mu(\Omega) = 1$ and $f \in L^\infty$ then

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p.$$

15. Show that $\int_{[0,\pi]} x^{-1/4} \sin x \, dm \le \pi^{3/4}$.