MTI Exercises 1: Solutions

1. For part (i). For any $x \in [a,b]$ we have that $x \in (a-1/n, b+1/n)$ for all $n \in \mathbb{N}$ and so $x \in \cap_{n=1}^{\infty}(a-1/n, b+1/n)$. On the other hand if $x \in \cap_{n=1}^{\infty}(a-1/n, b+1/n)$ then for all $n \in \mathbb{N}$ $a-x < 1/n$ and $x-b > 1/n$. Now if $a-x > 0$ then there exists $n \in \mathbb{N}$ such that $a-x > 1/n$ which would be a contradiction so $a-x \leq 0$ and thus $x \geq a$. Similarly we can show that $x \leq b$ which means $x \in [a,b]$.

   For part (ii). Fix $x \in (a,b)$. We know that $a < x < b$ and therefore there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \min\{(x-a),(b-x)\}$. Thus $a+1/n < x < b-1/n$ which means that $x \in \cup_{n\in\mathbb{N}}(a+1/n, b-1/n)$. On the other hand if $x \in \cup_{n\in\mathbb{N}}(a+1/n, b-1/n)$ then $a < x < b$ and so $x \in (a,b)$.

   Thus if we take a sigma algebra of subsets of $\mathbb{R}$ containing all open intervals then we can write any closed interval as a countable union of these elements so any closed interval must also be in the sigma-algebra.

2. To show that the Borel-sigma-algebra is generated by half-open intervals we need to show that the Borel sigma algebra ($\mathbb{B}$) contains all half open intervals and that the sigma algebra generated by half-open intervals contains all open intervals ($\mathbb{A}$). So let $a, b \in \mathbb{R}$ then $(a,b] = (a, b+1) \cap (b, b+1)^c$ thus $[a,b) \in \mathbb{B}$. On the other hand let $a, b \in \mathbb{R}$ we can write $(a,b) = \cup_{n\in\mathbb{N}}(a, b-(b-a)/(2n)]$ which means $(a,b) \in \mathbb{A}$, $(a,\infty) = \cup_{n=1}^{\infty}(a, a+n] \in \mathbb{A}$ and $(-\infty, a) = [a,\infty]^c$ where we have $[a,\infty] = \cup_n \in \mathbb{N}(a+1/n, \infty) \in \mathbb{A}$ and so $(-\infty, a) \in \mathbb{A}$.

   For the second part we proceed similarly, note that all half open rays are in the Borel-sigma algebra by definition. Let $a, b \in \mathbb{R}$ first note that
   $$(-\infty, a) = [a,\infty)^c = (\cup_{n=1}^{\infty}(a-1/n, \infty))^c$$
   so $(-\infty, a)$ is in the sigma-algebra generated by half open rays. We can then write $(a,b) = (a,\infty) \cap (-\infty, b)$ and so (a,b) is also in this sigma algebra.

3. It's clear that $\sigma(A) = \{\emptyset, \Omega, A, A^c\}$

4. Fix $A > 0$ and $\alpha \in \mathbb{R}$. Let
   $$A_\alpha := \{x : f_A(x) > \alpha\}$$
   and note that it suffices to show $A_\alpha \in \mathbb{X}$ for all $\alpha \in \mathbb{R}$. If $\alpha \geq A$ then $A_\alpha = \emptyset \in \mathbb{X}$ and if $\alpha < -A$ then $A_\alpha = X \in \mathbb{X}$. If $-A \leq \alpha < A$ then $f_k(x) > \alpha$ if and only if $f(x) > \alpha$ and so $A_\alpha = \{x : f(x) > \alpha\}$ which is measurable since f is measurable.

5. We can see straight away that (i) implies (ii) and by taking compliments that (ii) and (iii) ar equivalent. So we just need to prove that (ii) implies (i). So suppose (ii) holds, fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and note that since the rational numbers are dense in $\mathbb{R}$ that there exists a sequence of rational numbers α_n which converge to α from above. We can write

$$\{x : f(x) > \alpha\} = \cup_{n=1}^{\infty}\{x : f(x) > \alpha_n\}$$

and so $\{x : f(x) > \alpha\}$ is measurable. Thus f is $\mathbb{X}$-measurable.

6. Let $X = \mathbb{R}$ and $\mathbb{X} = \{\mathbb{R}, \emptyset\}$. Let $f(x) = 1$ if $x \geq 0$ and $f(x) = -1$ if $x < 0$. For all $x \in$
R $f^2(x) = |f| = 1$ and so is measurable. However $\{x \in \mathbb{R} : f(x) > 0\} = \mathbb{R}^+$ but $\mathbb{R}^+ \notin \mathbb{X}$ and so is not measurable.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone (for convenience we will assume it is increasing). Fix $\alpha \in \mathbb{R}$ and consider

$$A_\alpha := \{x \in \mathbb{R} : f(x) \geq \alpha\}.$$

If $A_\alpha = \emptyset$ then trivially A_α is measurable. If not then let $y = \inf\{x \in \mathbb{R} : f(x) \geq \alpha\}$. Since f is monotone increasing we have that $A_\alpha = (y, \infty)$ or $A_\alpha = [y, \infty)$ which are both measurable. Thus f is measurable.

8. Let

$$A_n = \{x : f_n(x) = 1\}.$$

If $\alpha \geq 1$ then $\{x : f_n(x) > \alpha\} = \emptyset\}$, if $0 \leq \alpha < 1$ then $\{x : f_n(x) > \alpha\} = A_n$ and if $\alpha < 0$ then $\{x : f_n(x) > \alpha\} = [0, 1]$. So we just need to show the sets A_n are measurable.

$$A_n = \bigcup_{(a_1,\dots,a_{n_1})\in\{0,1\}^{n-1}} \left(\left(\sum_{i=1}^{n-1} \frac{a_i}{2^i} \right) + \frac{1}{2^n}, \left(\sum_{i=1}^{n-1} \frac{a_i}{2^i} \right) + \frac{1}{2^{n-1}} \right].$$

So the set A_n can be written as a union of half-open intervals, which by question 4 we know are Borel measurable. So A_n is Borel measurable and thus f_n is measurable.

9. Recall from the lectures we fix $n \in \mathbb{N}$ and for $k = 0, 1, \dots, 2^n n - 1$ define

$$E_{k,n} = \left\{ x \in X : f(x) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) \right\}.$$

We also define

$$E_{2^n n,n} = \{x : f(x) \geq n\}$$

and let $\phi_n(x) = \frac{k}{2^n}$ if $x \in E_{k,n}$. Now since in this case $0 \leq f(x) \leq K$ for all $x \in X$ if we take $n > K$ then $E_{2^n n,n} = \emptyset$. Thus if we let $\epsilon > 0$

and choose $N \geq \max\{K, -\frac{\log\epsilon}{\log 2}\}$ then for all $x \in X$ $x \in E_{k,n}$ where $k < 2^n n$. So

$$|f(x) - \phi_n(x)| \leq \frac{1}{2^n} \leq \epsilon.$$

Thus the convergence is uniform.

10. The first two points are trivial. To show (iii) we use $E \setminus F = E \cap F^c$ and show first that $f^{-1}(F^c) = (f^{-1}(F))^c$. It follows from the identity

$$\begin{aligned} f^{-1}(F^c) = \{x \in \Omega_1 : f(x) \in F^c\} &= \{x \in \Omega_1 : f(x) \notin F\} \\ &= (\{x \in \Omega_1 : f(x) \in F\})^c = (f^{-1}(F))^c. \end{aligned}$$

Now we want to show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ for any $A, B \subset \Omega_1$. We use the following argument. We know that $x \in f^{-1}(A \cap B)$ is equivalent to $f(x) \in A \cap B$, which is equivalent to $f(x) \in A$ and $f(x) \in B$, which is equivalent to $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$, equivalent to $x \in f^{-1}(A) \cap f^{-1}(B)$. So we are done.

Now we have to show point (iv). By definition we have

$$\begin{aligned} x \in f^{-1}(\cup_\alpha E_\alpha) \Leftrightarrow x \in \{x \in \Omega_1 : f(x) \in \cup_\alpha E_\alpha\} &\Leftrightarrow \exists \alpha : \ x \in \{x \in \Omega_1 : f(x) \in E_\alpha\} \\ \Leftrightarrow \ x \in \cup_\alpha \{x \in \Omega_1 : f(x) \in E_\alpha\} &\Leftrightarrow x \in \cup_\alpha f^{-1}(E_\alpha). \end{aligned}$$

The last property can be shown using De Morgan's laws.

11. Use previous exercise.

12. Use previous exercise.