MTI Exercises 2: Solutions

1. We have to check properties of a measure. It's clear that $\lambda(\emptyset) = \mu(\emptyset) = 0$, $\lambda(E) = \mu(A \cap E) \geq 0$ for any $E \in \mathfrak{U}$. Now we have to check that if $E_i \in \mathfrak{U}$ $(i = 1, 2, ...)$ are disjoint then $\lambda(\cup_i E_i) = \sum_i \lambda(E_i)$. By definition of λ we have

$$\lambda(\cup_i E_i) = \mu(A \cap (\cup_i E_i)) = \mu(\cup_i (A \cap E_i)) = \sum_i \mu(A \cap E_i) = \sum_i \lambda(E_i).$$

2. We can use induction. We know that $a_1\mu_1$ is a measure so the base case is true. Now suppose $\nu_k = a_1\mu_1 + \ldots + a_k\mu_k$ is a measure and consider $\nu_{k+1} = a_1\mu_1 + \ldots + a_{k+1}\mu_{k+1}$. We know that

$$\nu_{k+1}(\emptyset) = \nu_k(\emptyset) + a_{k+1}\mu_{k+1}(\emptyset) = 0.$$

For any $A \in \mathbb{X}$ we have that

$$\nu_{k+1}(A) = \nu_k(A) + a_{k+1}\mu_{k+1}(A) \geq 0 + 0 = 0.$$

Finally let $A_1, A_2, \ldots \in X$ be disjoint

$$\begin{aligned} \nu_{k+1}\left(\cup_{i=1}^{\infty} A_i\right) &= \nu_k\left(\cup_{i=1}^{\infty} A_i\right) + a_{k+1}\mu_{k+1}\left(\cup_{i=1}^{\infty} A_i\right) \\ &= \sum_{i=1}^{\infty} \nu_k(A_i) + \sum_{i=1}^{\infty} a_{k+1}\mu_{k+1}(A_i) \\ &= \sum_{i=1}^{\infty} \nu_{k+1}(A_i) \end{aligned}$$

therefore if ν_k is a measure then so is ν_{k+1} and the result follows by induction.

3. To show that μ is a measure you just check the properties of measure and it is straightforward. In the other direction, if μ is any measure on $\mathbb{N}$ then you restrict μ to $\{n\}$ for all $n \in \mathbb{N}$, construct a sequence $\{a_n\}$ and new measure ν, using this sequence $\{a_n\}$ exactly as in the formulation of the question. This new measure ν coincides with μ.

4. This is not a measure. To see this take $E_n = \{n\}$ note that $\cup_{n=1}^{\infty} E_n = \mathbb{N}$ and note that this union is disjoint. However $\mu(\mathbb{N}) = \infty$ but $\sum_{n=1}^{\infty} \mu(E_n) = 0$ and thus μ is not a measure.

5. We know $A \subset C$, $B \subset C$ and $\mu(A) = \mu(C)$. Therefore $\mu(C \setminus A) = 0$. It is clear that $\mu(A \cap B) + \mu((C \setminus A) \cap B) = \mu(C \cap B) = \mu(B)$. Recalling $\mu(C \setminus A) = 0$ we get the result.

6. Take $F_n = (n, +\infty)$ it is clear that $\cap_n F_n = \emptyset$ but $\mu(F_n) = \infty$ for all n. Result follows.

7. For the first part let $B_n = \cap_{m=n}^{\infty} A_m$ and note that $B = \cup_{n=1}^{\infty} B_n$ and that $B_n \subset B_{n+1}$. Thus

$$\mu(B) = \lim_{n\to\infty} \mu(B_n) = \liminf_{n\to\infty} \mu(B_n)$$

and $B_n \subset A_n$ so $\mu(B_n) \leq \mu(A_n)$ which means that $\mu(B) \leq \liminf_{n\to\infty} \mu(A_n)$. For the second part let $B_n = \cup_{n=m}^{\infty} A_n$ and note that $A = \cap_{n=1}^{\infty} B_n$. Since $\mu(B_1) < \infty$ and for all n we have that $B_n \supseteq B_{n+1}$ we know

$$\mu(A) = \lim_{n\to\infty} \mu(B_n) = \limsup_{n\to\infty} \mu(B_n).$$

However $B_n \supseteq A_n$ and so $\mu(B_n) \geq \mu(A_n)$ which means that $\mu(A) \geq \limsup_{n\to\infty} \mu(A_n)$. We now show that this inequality may fail if $\mu(\cup_{n=1}^{\infty} A_n) = \infty$. For example consider the measure space $(R, \mathbb{B}, \lambda)$ and the sets $A_n = [n-1, n]$. We have that $\lambda(\limsup_{n\to\infty} A_n) = \lambda(\emptyset) = 0$ but $\limsup_{n\to\infty} \lambda(A_n) = \limsup_{n\to\infty} 1 = 1$.

8. Take a trivial σ-algebra consisting of Ω and $\emptyset$ and define a measure to be $\mu(\Omega) = \mu(\emptyset) = 0$. It's clear that this measure space is not complete.

9. Either use additivity of the integral or a put ϕ is a standard form by constructing disjoint sets.

10. Trivial.

11. Borel measurbility of f follows directly from the definition (consider sets $A_\alpha = \{x : f(x) > \alpha\}$). To compute the integral we notice that

$$\int_0^1 f d\mu = \sum_{n=1}^{\infty} n \frac{2^{n-1}}{3^n} = 3$$

12. Let $\epsilon > 0$ and let $N \geq \frac{1}{\epsilon}$. We have that for $n \geq N$

$$\sup_{x\in\mathbb{R}}\{|f_n(x) - f(x)|\} = \sup_{x\in\mathbb{R}}\{f_n(x)\} \leq \frac{1}{n} \leq \epsilon.$$

So $f_n \to 0$ uniformly. However

$$\int f_n \mathrm{d}\lambda = \frac{1}{n}\lambda([0, n]) = 1$$

and $\int f \mathrm{d}\lambda = 0$. This does not contradict the Monotone convergence theorem since $f_1(1) = 1 < f_2(1) = 1/2$ which means that $f_n(x)$ is not a monotone increasing sequence for all values of x. Yes Fatou's Lemma does apply, we have

$$0 = \int \liminf_{n\to\infty} f_n(x)\mathrm{d}\lambda \leq 1 = \liminf \int f_n(x)\mathrm{d}\lambda.$$

13. Fix $\epsilon > 0$ and choose N such that for all $n \geq N$ we have that $\sup_{x\in X} |f_n(x) - f(x)| \leq \epsilon/\mu(X)$. We have that for $n \geq N$

$$\int f_n \mathrm{d}\mu \leq \int (f + \epsilon/\mu(X))\mathrm{d}\mu$$

which means that since $\int \epsilon/\mu(X)\mathrm{d}\mu(x) = \epsilon$

$$\int f_n \mathrm{d}\mu \leq \int f \mathrm{d}\mu + \epsilon.$$

We also have

$$\int f \mathrm{d}\mu \leq \int (f_n + \epsilon/\mu(X))\mathrm{d}\mu$$

which means that

$$\int f_n \mathrm{d}\mu \geq \int f \mathrm{d}\mu - \epsilon.$$

The result follows.