MTI Exercises 6: Solutions

1. Take a sequence $f_n(x) = 1$ on $(n, n+1)$ and 0 otherwise. Clearly there is a.e. convergence but no convergence in measure.

2. See Corollary 7.6. in the lecture notes.

   Second part. Let $a, \epsilon > 0$ and note we can fix N such that for all $n > N$ we have that

   $$\mu\{x : |f_n(x) - f(x)| > a/2\} < \epsilon/2.$$

   By the triangle inequality if we take $n, m > N$ then

   $$\{x : |f_n(x) - f_m(x)| > a\} \subset \{x : |f(x) - f_n(x)| > a/2\} \cup \{x : |f(x) - f_m(x)| > a/2\}.$$

   Thus

   $$\mu(\{x : |f_n(x) - f_m| > a\}) < \epsilon.$$

   It follows that f_n is Cauchy in measure.

3. a) Let $c > 0$. We have that by the triangle inequality

   $$\begin{aligned}&\{x : |af_n(x) + bg_n(x) - af(x) - bg(x)| > c\}\\ &\subset \{x : |a||f(x) - f_n(x)| > c/2\} \cup \{x : |b||g(x) - g_n(x)| > c/2\}.\end{aligned}$$

   However since $f_n \to f$ in measure and $g_n \to g$ in measure we can see that

   $$\lim_{n\to\infty} \mu(\{x : |a||f(x) - f_n(x)| > c/2\}) = \lim_{n\to\infty} \mu(\{x : |b||g(x) - g_n(x)| > c/2\}) = 0.$$

   Thus

   $$\lim_{n\to\infty} \mu(\{x : |af_n(x) + bg_n(x) - af(x) - bg(x)| > c\}) = 0.$$

   b) We have $||f_n(x)| - |f(x)|| \leq |f_n(x) - f(x)|$ and therefore for any $a > 0$

   $$\{x : ||f_n(x)| - |f(x)|| > a\} \subseteq \{x : |f_n(x) - f(x)| > a\}$$

   and the result immediately follows since then

   $$\mu(\{x : ||f_n(x)| - |f(x)|| > a\}) \leq \mu(\{x : |f_n(x) - f(x)| > a\}) \to 0 \text{ as } n \to \infty.$$

4. We observe $\{|f_n g_n - fg| > 3\epsilon\} \subset \{|f_n - f||g_n - g| > \epsilon\} \cup \{|f_n - f||g| > \epsilon\} \cup \{|f||g_n - g| > \epsilon\}$. For any set A we have

$$\mu(\{|f_n g_n - fg| > 3\epsilon\}) \leq \mu(A^c) + \mu(\{|f_n - f||g_n - g| > \epsilon\} \cap A) \\ +\mu(\{|f_n - f||g| > \epsilon\} \cap A) + \mu(\{|f||g_n - g| > \epsilon\} \cap A)$$

It's clear that for every $\delta > 0$ there exists $N > 0$ such that $\mu(\{|f| > N\}) + \mu(\{|g| > N\}) < \delta$. We take

$$A = \{|f| \leq N\} \cap \{|g| \leq N\}.$$

It is now clear that

$$\mu(\{|f_n g_n - fg| > 3\epsilon^2\}) \leq \delta + \mu(\{|f_n - f||g_n - g| > \epsilon\}) \\ + \mu(\{|f_n - f| > \epsilon/N\}) + \mu(\{|g_n - g| > \epsilon/N\}).$$

The claim follows.

5. i. Let $A \in X$ such that for all $x \in A$ we have that $\lim f_n(x) = f(x)$ and $\mu(A^c) = 0$. Therefore $\lim_{n\to\infty} f_n\chi_A = f\chi_A$ and so by the Standard Fatou's lemma and the fact that $\mu(A^c) = 0$

$$\int f\mathrm{d}\mu = \int f\chi_A\mathrm{d}\mu \leq \liminf_{n\to\infty} \int f_n\chi_A\mathrm{d}\mu = \liminf_{n\to\infty} \int f_n\mathrm{d}\mu.$$

ii. Take a subsequence g_k of f_n such that $\lim_{k\to\infty} \int g_k\mathrm{d}\mu = \liminf_{n\to\infty} \int f_n\mathrm{d}\mu$. We then know that g_k converges to f in measure and so by Theorem 7.5 there exists a subsequence h_l of g_k such that $\lim h_l(x) = f(x)$ μ almost everywhere. Therefore by the previous part

$$\liminf_{n\to\infty} \int f_n\mathrm{d}\mu = \lim_{l\to\infty} \int h_l\mathrm{d}\mu \geq \int f\mathrm{d}\mu.$$

6. Suppose f_n converges in measure to f and $|f_n| \leq g$ for all n where $g \in L(X)$. We can find a subsequence g_k of f_n such that g_k tends to f μ almost everywhere so the standard Dominated Convergence Theorem tells us that $f \in L$. Now we suppose that $\lim \int f_n\mathrm{d}\mu \neq \int f\mathrm{d}\mu$ and so we can find an $\epsilon > 0$ and a subsequence g_k of f_n such that $\left|\int f_n\mathrm{d}\mu - \int f\mathrm{d}\mu\right| > \epsilon$ for all n. However g_k converges to f in measure so we can find a subsequence of g_k, h_l for which $\lim_{l\to\infty} h_l(x) = f(x)$ for μ a.e. x and $|h_l| \leq g$. Therefore by the standard dominated convergence theorem we have that $\lim_{l\to\infty} \int h_l\mathrm{d}\mu = \int h\mathrm{d}\mu$ but this is a contradiction.

7. Fix $a, \epsilon > 0$ we can find N such that for $n > N$ we have that the set

$$A := \{x : |f_n(x) - f(x)| > a\}$$

satisfies $\mu(A) \leq \epsilon$. Thus for $n > N$ we can write

$$r(f_n - f) = \int \frac{|f_n - f|}{1 + |f_n - f|} \mathrm{d}\mu \leq \int_{A^c} a \mathrm{d}\mu + \int_A 1 \mathrm{d}\mu$$

and so

$$r(f_n - f) \leq a\mu(X) + \epsilon.$$

The result follows since a, ϵ can be chosen arbitrarily small.

Now if $\lim_{n\to\infty} r(f_n - f)$ then $\int |f_n - f| \mathrm{d}\mu = 0$ and so we can see that f_n converges in measure to f.

8. If $f_n \to f$ in L^p then $\{f_n\}$ is Cauchy in L^p, meaning that $\|f_n - f_m\| \to 0$ as $n, m \to \infty$. Therefore, we have $\|f_n - f_{n_k}\| \to 0$ as $n, k \to \infty$. Since $f_{n_k} \to g$ in L_p we have that $\|f - g\| = 0$.

9. Example was given in lectures. Take a cyclic sequence on $[0, 1]$: $f_1(x) = \chi_[0, 1/2]$, $f_2(x) = [1/2, 1]$, $f_3(x) = [0, 1/3]$, $f_4(x) = [1/3, 2/3]$, $f_5(x) = [2/3, 1]$, ...

10. Fix $\alpha > 0$. We then let $A = \{x \in X : |Y(x) - \mu| \geq \alpha\}$. We then have that

$$\sigma^2 = \int |Y - \mu|^2 \mathrm{d}\mathbb{P} \geq \int_A |Y - \mu|^2 \mathrm{d}\mathbb{P} \geq \alpha^2 \mathbb{P}(A).$$

Thus $\mathbb{P}(A) \leq \frac{\sigma^2}{\alpha^2}$ which is exactly what we needed to prove.