

Lecture notes¹ for Applied Partial Differential Equations 2 (MATH20402)

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¹This is a revised version of 2014 notes. Modified by Dr Valeriy Slastikov.

Chapter 1

Introduction and Review

Many physical systems are modelled in terms of continuous functions (e.g. temperature, velocity) and depend upon more than one variable (e.g. space and time).

The mathematical equations that describe the variations of these functions are nearly always described by rates/gradients (i.e. involve derivatives) and therefore lead to **partial differential equations** (PDEs).

Essentially the whole subject of applied mathematics is underpinned by PDEs !

PDEs are used to describe: fluid dynamics, elasticity, radio communications, chemical reactions, climate modelling, stock markets, and so on...

Many of these physical systems are complicated and lead to problems which are difficult to model mathematically. Many reduce under certain simplifying assumptions to more basic PDEs. Our main focus is to consider a simple important set of examples of PDEs which we shall derive from simple physical models; however, these PDEs are often closely related to those describing more complex physical systems.

The course concentrates on describing analytical methods for solving these PDEs. Note: often, but not always, life is not as simple as we make out here and numerical methods are needed to help solve PDEs.

Before looking at PDEs, it is helpful to review some important concepts of ordinary differential equations (ODEs).

1.1 Review of ODEs

An ODE for a function of one variable, $u(x)$ say, is an equation involving $x, u(x), u'(x), u''(x), \dots$, which holds in some interval of x , which may be finite or infinite.

Defn: Here u depends on x and so u is called the **dependent variable** and x is called the **independent variable**.

1.1.1 Types of ODE

Defn:

- (i) The **order** of an ODE is the value of the highest derivative.
- (ii) An n th order ODE for $u(x)$ is **linear** if it can be written in the form

$$a_n(x)u^{(n)}(x) + \dots + a_2(x)u''(x) + a_1(x)u'(x) + a_0u(x) = f(x)$$

for some $a_n(x)$. Otherwise **non-linear**.

- (iii) A linear ODE is said to be **homogeneous** if $u(x) = 0$ is a solution. I.e. $f(x) = 0$ in above.

1.1.2 Examples and Terminology

1. Newton's Law of cooling. Temperature $T(t)$ satisfies the ODE:

$$\frac{dT}{dt} = -k(T - \bar{T}) \quad \text{or} \quad T' + kT = k\bar{T}$$

for time $t > 0$ where k (rate of cooling) and $\bar{T}$ (ambient temperature) are constants.

Here, T is the dependent variable, t is independent variable.

The ODE is: 1st order, linear and non-homogeneous.

Solution (separating variables/integrating factor):

$$T(t) - \bar{T} = Ce^{-kt}, \quad C \text{ constant}$$

If given **initial condition** (IC) $T(0) = T_0$, then $C = T_0 - \bar{T}$.

Defn: The ODE plus IC define an **initial-value problem**.

2. Consider

$$u''(x) - 3u'(x) + 2u(x) = 2x$$

ODE is: 2nd order, linear, inhomogeneous.

Dependent variable is u , independent variable x .

Solution (Complementary Function and Particular Solution):

$$u(x) = x + \frac{3}{2} + Ae^x + Be^{2x}$$

where A and B are arbitrary constants.

Note: In general, ODEs have **general solutions** and number of arbitrary constants equals to the order of the ODE.

If ODE holds over $a < x < b$, say, then we often specify information about the solution at the two end points, $x = a$, $x = b$, say, $u(a) = 0$, $u(b) = 1$. These are called **boundary conditions** (BCs).

Defn: The ODE plus BCs define a **(two-point) boundary-value problem**

1.1.3 Linear Homogeneous Problems

Want to focus on ODEs which will be important later. First

Defn: A BC at $x = a$ is called **homogeneous** if it is of the form

- $u(a) = 0$ (**homogeneous Dirichlet BC**)
- $u'(a) = 0$ (**homogeneous Neumann BC**)
- $\alpha u(a) + \beta u'(a) = 0$, given constants α, β (**homogeneous Robin BC**)

If non-zero on RHS then BC is called **inhomogeneous**.

Defn: A **Homogeneous Problem** comprises a homogeneous linear ODE with homogeneous BCs.

Obviously, $u(x) \equiv 0$ is always a solution of such a problem, (the **trivial solution**) but is this always the only solution ...

1.1.4 Key illustrative example

Consider the homogeneous problem

$$u''(x) - \lambda u(x) = 0, \quad 0 < x < 1, \quad \text{with Dirichlet BCs} \quad u(0) = u(1) = 0$$

How do the solutions depend on λ ?

Usual method of solution: Try $u = Ae^{rx}$ gives characteristic equation

$$r^2 - \lambda = 0, \quad \text{so} \quad r = \pm\sqrt{\lambda}$$

Three different types of solution depending on $\lambda > 0$ (real roots) $\lambda = 0$ (repeated roots) and $\lambda < 0$ (complex conjugate roots)

Case 1: $\lambda > 0$. Helps to write $\lambda = \mu^2$. Then $r = \pm\mu$ and the general solution can be written

$$u(x) = Ce^{\mu x} + De^{-\mu x},$$

or (better for finite domains), combine exponentials

$$u(x) = A \cosh(\mu x) + B \sinh(\mu x).$$

Apply BC $u(0) = 0$ gives $A = 0$, and then BC $u(1) = 0$ gives $B \sinh \mu = 0$. Since $\mu > 0$, must have $B = 0$. So only solution is $u(x) \equiv 0$.

Case 2: $\lambda = 0$. Now ODE is $u''(x) = 0$ so general solution is $u(x) = Ax + B$. Fitting BCs implies $A = B = 0$ and $u(x) \equiv 0$ again.

Case 3: $\lambda < 0$. Helps to write $\lambda = -k^2$. Then $r = \pm ik$ and general solution is

$$u(x) = Ce^{ikx} + De^{-ikx}$$

or (again, better for finite intervals)

$$u(x) = A \cos(kx) + B \sin(kx)$$

Apply BC $u(0) = 0$ to give $A = 0$ and BC $u(1) = 0$ to give $B \sin k = 0$. **Now:**

- If $k \neq n\pi, n \in \mathbb{Z}$ then $\sin k \neq 0$, so $B = 0$ and only solution is $u(x) \equiv 0$.
- If $k = n\pi, n \in \mathbb{Z}$ then $u(x) = B_n \sin(n\pi x)$ is a solution $\forall B_n$ and $\forall n \in \mathbb{Z}$

1.1.5 Eigenvalues and eigenfunctions

Summary: we have seen that the linear homogeneous problem

$$u''(x) = \lambda u(x) \quad 0 < x < 1 \quad \text{with } u(0) = u(1) = 0 \quad (1.1)$$

has

- zero solutions for most values of λ ,
- non-zero solutions when $\lambda = -n^2\pi^2, n = 1, 2, \dots$

Note: $n = -1, -2, \dots$ duplicates same values of λ so not needed.

For any of these values of λ , there are infinitely many solutions (B_n is an arbitrary constant multiplier of $\sin(n\pi x)$).

This is reminiscent of linear algebra where a matrix A has eigenvalues given by $Ax = \lambda x$ and each eigenvalue has an eigenvector which can be multiplied by an arbitrary constant.

Here, we can think of u'' as $\mathcal{L}u$ where $\mathcal{L}$ is the **operator** (here it is d^2/dx^2) which transforms u into u'' .

By analogy we write (1.1) as

$$\mathcal{L}u = \lambda u, \quad \text{where } \mathcal{L} = d^2/dx^2 \text{ is a **differential operator**}$$

and we call $\lambda = \lambda_n = -n^2\pi^2, n = 1, 2, \dots$ **eigenvalues**.

For each eigenvalue, the non-trivial solutions are called **eigenfunctions**. Here, the eigenfunctions corresponding to $\lambda = -n^2\pi^2$ are $u(x) = u_n(x) = \sin n\pi x$.

Note: any multiple of an eigenfunction is also an eigenfunction.

Note: The ODE alone does not determine the eigenvalues; the BCs are essential.

1.1.6 Principle of linear superposition

For a linear homogeneous problem any linear combination of solutions is also a solution.

Proof: $u(x) = u_1(x)$ and $u(x) = u_2(x)$ both satisfy the same homogeneous ODE:

$$a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x) = 0 \quad (1.2)$$

and the same homogeneous BCs then so does $u(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x)$ for any arbitrary constants α_1 and α_2 (easy to confirm).

Obvious to extend: if $u = u_n(x)$ satisfies (1.2) for $n = 1, 2, \dots, N$, then so does

$$u(x) = \sum_{n=1}^N \alpha_n u_n(x)$$

for arbitrary constants α_n .

1.1.7 Inhomogeneous problems

Either an inhomogeneous ODE *or* a homogeneous ODE with inhomogeneous BCs. Possibly both.

Now the principle of superposition does **not** hold

Examples:

- (i) Let u_1 and u_2 be two solutions satisfying

$$u''(x) - u(x) = 0 \quad 0 < x < 1 \quad \text{with inhomogeneous BCs } u(0) = 0, u(1) = 1.$$

Now $u = \alpha_1 u_1 + \alpha_2 u_2$ satisfies ODE and $u(0) = 0$ but $u(1) = \alpha_1 + \alpha_2 \neq 1$ for arbitrary α_1 and α_2 .

- (ii) Similarly consider inhomogeneous ODE: $u''(x) - \lambda u(x) = f(x)$, $0 < x < 1$ with $u(0) = u(1) = 0$.

1.1.8 Reducing inhomogeneous problems to homogeneous problems

Trick is to find a **particular solution** satisfying the inhomogeneous problem (similar to C.F. + P.S.)

In e.g. (i) above, observe that $u_p(x) = \sinh(x) / \sinh(1)$ is a particular solution of the ODE with the inhomogeneous BCs.

Define $v(x) = u(x) - u_p(x)$ and now $v(x)$ satisfies the homogeneous problem:
 $v''(x) - v(x) = 0$, $v(0) = 0$, $v(1) = 0$.

This will be useful later ...

1.2 Introduction to PDEs

1.2.1 What is a PDE ?

It's an equation for a function of $n \geq 2$ variables, involving partial derivatives.

As in ODEs it must hold in a region, $\mathbb{R}^n$ of space, which can be finite or infinite.

Notation:

$$\frac{\partial f}{\partial x} \equiv f_x \equiv \partial_x f, \quad \frac{\partial^2 f}{\partial x^2} \equiv f_{xx} \equiv \partial_{xx} f, \quad \frac{\partial^2 f}{\partial x \partial y} \equiv f_{xy} \equiv \partial_{xy} f, \text{ etc.}$$

Note: For any 'well-behaved' $f(x, y)$,

$$f_{xy} \equiv f_{yx}$$

We always assume well-behaved: I.e. continuous and differentiable as often necessary.

As in ODEs, for a function $u(x, y)$, u is called the **dependent variable** and x, y are the **independent variables**.

1.2.2 Types of PDE

The terminology parallels exactly that for ODEs. I.e.

- The **order** is the value of the highest derivative (u_{xy} is a 2nd derivative)
- A **linear** PDE is one in which the dependent variable and its derivatives occur linearly. Otherwise **non-linear**.
- A **homogeneous** PDE is one for which $u = 0$ is a solution.

1.2.3 Examples

1. $u_x = 2x \sin y + e^{xy}$ for $u(x, y)$.

PDE is: 1st order, linear & inhomogeneous.

Solution is simple: integrating gives

$$u(x, y) = x^2 \sin y + \frac{e^{xy}}{y} + f(y)$$

where f is **any** real function.

Note: Whereas ODEs have *general solutions* involving *arbitrary constants*, PDEs have general solutions with *arbitrary functions* (generally as many as the order of the PDE).

2.

$$\phi_{\xi\eta} = 0$$

PDE for $\phi(\xi, \eta)$ is: 2nd order, linear & homogeneous.

General solution found by successively integrating. First integration w.r.t. η gives $\phi_{\xi} = h(\xi)$ for arbitrary function h . Then integrate w.r.t. ξ to give

$$\phi(\xi, \eta) = \int^{\xi} h(t)dt + g(\eta) = f(\xi) + g(\eta)$$

where f is arbitrary as h was arbitrary.

3.

$$\phi_t = \phi_{xx}$$

PDE for $\phi(x, t)$ is: 2nd order, linear & homogeneous.

Called **diffusion equation**.

A **particular solution** is $\phi(x, t) = 2at + bx + ax^2$ for constants a, b . But *not* a general solution.

4.

$$\rho_t + \rho\rho_x = 0.$$

PDE for $\rho(x, t)$ is: 1st order, non-linear & homogeneous.

No obvious solution ... later...

5. Electric field due to static charges governed by

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 4\pi\rho$$

where $\rho(x, y, z)$ is given charge density. Called **Poisson's equation**.

PDE for $\phi(x, y, z)$ is linear, 2nd order, inhomogeneous.

(**Online Extra:** Given electric field $\mathbf{E}$, satisfying time-independent Maxwell's equations, $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{E} = 4\pi\rho$, it follows from the first equation that $\mathbf{E} = \nabla\phi$ where ϕ is scalar function and using this in the second equation gives Poisson's equation)

1.3 Classification of linear second order PDEs

The general form of the second order linear PDE is

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad (1.3)$$

The classification of PDEs is similar to classification of the quadratic equation of conic sections, where equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents hyperbola, parabola or ellipse depending on sign of $B^2 - 4AC$.

Let us derive canonical forms of second order linear PDEs with constant coefficients. We have to look only at the second order terms

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y) \quad (1.4)$$

We define the following operator

$$L = A\partial_{xx} + B\partial_{xy} + C\partial_{yy} = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

Then we can rewrite (1.4) as $Lu = H$. If we want to put this equation in a canonical form we have to diagonalize the matrix

$$\mathcal{A} = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}.$$

There are three distinct cases for eigenvalues λ_1 and λ_2 depending on determinant of $\mathcal{A}$ ($\det \mathcal{A} = AC - B^2/4$):

1. if $\det \mathcal{A} < 0$ then λ_1 and λ_2 have different signs (say $\lambda_1 > 0$ and $\lambda_2 < 0$) and $Lu = |\lambda_1|u_{\xi\xi} - |\lambda_2|u_{\eta\eta}$ – **hyperbolic equation**;
2. if $\det \mathcal{A} = 0$ then one of the eigenvalues is zero (say $\lambda_1 = 0$) and $Lu = \lambda_2 u_{\eta\eta}$ – **parabolic equation**;
3. if $\det \mathcal{A} > 0$ then λ_1 and λ_2 have the same signs (say $\lambda_1 > 0$ and $\lambda_2 > 0$) and $Lu = |\lambda_1|u_{\xi\xi} + |\lambda_2|u_{\eta\eta}$ – **elliptic equation**.

Here (ξ, η) are new coordinates found by a linear transformation of (x, y) dictated by $\mathcal{A}$. When coefficients A, B, C depend on (x, y) the classification is done in the same way however it depends on the specific point (x, y) under consideration.

Chapter 2

Derivation of PDEs

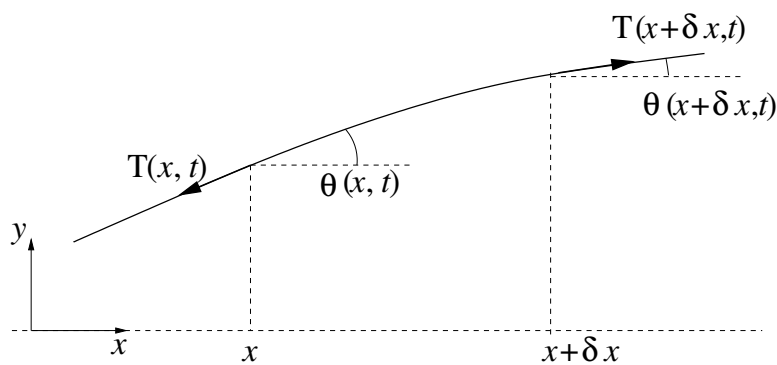
We introduce three main equations which we study in detail in this course. They are 2nd order, linear PDEs.

2.1 The 1D wave equation

Describes waves on a string, sound in a pipe etc, compressive elastic waves in rod, surface water waves in a channel etc...

Derivation below is based on waves on a string.

Take a taut string of constant density (mass per unit length) ρ . It's small displacement is $y = u(x, t)$ (we assume that displacement is small and neglect the horizontal component of displacement).



The forces acting on a small piece of the string δx are tension T and gravity $-\rho g \delta x$.

We want to balance the forces. There is no motion in the horizontal direction and therefore we have

$$T(x, t) \cos \theta(x, t) = T(x + \delta x, t) \cos \theta(x + \delta x, t).$$

Taking $\delta x \rightarrow 0$ we obtain

$$\frac{\partial}{\partial x} (T(x, t) \cos \theta(x, t)) = 0$$

and therefore

$$T(x, t) \cos \theta(x, t) = T_0(t).$$

In the vertical direction we apply **Newton's Law** to the section of length δx , mass $\rho \delta x$ to obtain

$$\rho \delta x \frac{\partial^2 u}{\partial t^2} = T(x + \delta x, t) \sin \theta(x + \delta x, t) - T(x, t) \sin \theta(x, t) - \delta x \rho g.$$

Dividing both parts by $\rho \delta x$ and taking $\delta x \rightarrow 0$ we obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho} \frac{\partial}{\partial x} (T(x, t) \sin \theta(x, t)) - g.$$

Using equation for $T(x, t)$ from the horizontal force balance we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0(t)}{\rho} \frac{\partial}{\partial x} (\tan \theta(x, t)) - g.$$

By geometrical reasons it is clear that $\theta(x, t) = \frac{\partial u(x, t)}{\partial x}$ and therefore assuming $T_0(t) \equiv T_0$ we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho} \frac{\partial^2 u}{\partial x^2} - g.$$

We can also add other external forces to yield

$$u_{tt} = c^2 u_{xx} + f(x, t), \quad \text{where } c = \sqrt{T/\rho}.$$

Defn: c is an important parameter – **wave speed** (see later).

The 2D/3D wave equation

Given by

$$u_{tt} = c^2 \nabla^2 u + f(x, t)$$

where, in 2D, the Laplacian is

$$\nabla^2 u = u_{xx} + u_{yy}$$

and describes the vibrations of a membrane or the ripples on a pond (for example); in 3D,

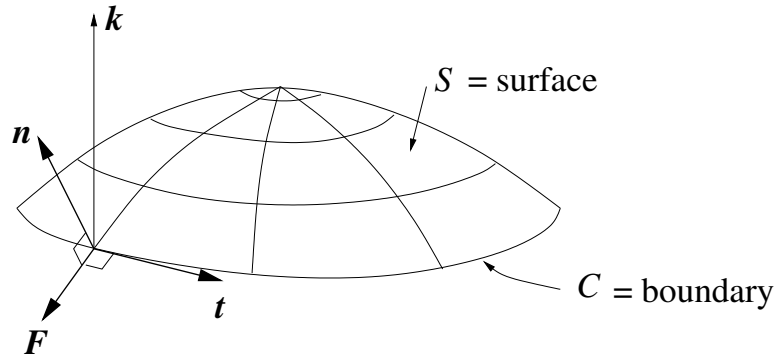
$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$$

and describes sound waves in air, radio waves etc...

2.2 Online extra: The 2D wave equation (membrane equation)

Appears in many physical applications, derived here from considering a taut membrane.

Consider a small arbitrary section of an elastic membrane, displaced from rest position $z = 0$ to $z = u(x, y, t)$. Assume constant tension T and ρ is mass per unit area.



Force is $\mathbf{F} = T\mathbf{t} \times \mathbf{n}$ is the force acting on C .

Vertical component of force around C is $\mathbf{F} \cdot \mathbf{k}$

Application of Newton's Law on S in vertical direction is

$$\int_S \rho \frac{\partial^2 u}{\partial t^2} dt = \oint_C \mathbf{F} \cdot \mathbf{k} ds$$

(Balance the total integrated mass times acceleration with the net force, coming from the boundary)

Now, $\mathbf{F} \cdot \mathbf{k} = T(\mathbf{t} \times \mathbf{n}) \cdot \mathbf{k} = T(\mathbf{n} \times \mathbf{k}) \cdot \mathbf{t}$

Next, apply **Stokes' theorem**:

$$\oint_C T(\mathbf{n} \times \mathbf{k}) \cdot \mathbf{t} ds = \int_S T[\nabla \times (\mathbf{n} \times \mathbf{k})] \cdot \mathbf{n} dS$$

We note that (geometrically)

$$\mathbf{n} = \frac{(-u_x \mathbf{i} - u_y \mathbf{j} + \mathbf{k})}{\sqrt{u_x^2 + u_y^2 + 1}}$$

and it is assumed that $u_x, u_y \ll 1$ so

$$\mathbf{n} \approx (-u_x \mathbf{i} - u_y \mathbf{j} + \mathbf{k})$$

Then

$$(\mathbf{n} \times \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -u_x & -u_y & 1 \\ 0 & 0 & 1 \end{vmatrix} = -u_y \mathbf{i} + u_x \mathbf{j}$$

and so

$$\nabla \times (\mathbf{n} \times \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -u_y & u_x & 0 \end{vmatrix} = (u_{xx} + u_{yy})\mathbf{k} \equiv (\nabla^2 u)\mathbf{k}$$

Finally, using definition of $\mathbf{n}$,

$$\mathbf{n} \cdot [\nabla \times (\mathbf{n} \times \mathbf{k})] \equiv (\nabla^2 u)$$

and insert into integrals over S to get

$$\int_S \rho \frac{\partial^2 u}{\partial t^2} dS = \int_S T \nabla^2 u dS.$$

Must be true for *all* arbitrary sections S of the membrane, so has to be that

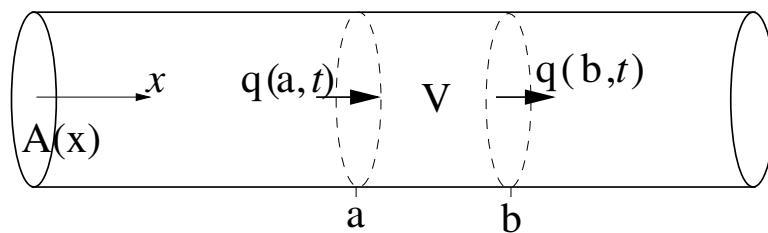
$$u_{tt} = c^2 \nabla^2 u, \quad \text{where } c^2 = T/\rho$$

This is the **2D wave equation**... The **3D wave equation** (e.g. sound waves) is given by the same equation, but with

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$$

2.3 The 1D diffusion/heat equation

Describes, for e.g. conduction of heat in a rod, spreading of chemicals in a tube.



We'll set the derivation in a general context.

So let $\rho(x, t)$ be the 'density' of a species in a tube, of generally varying cross-sectional area $A(x)$. Then total 'mass' of the species in V is

$$E = \int_a^b A(a) \rho(x, t) dx$$

Let $q(x, t)$ represent the 'mass' flux (flow rate) of species across position x in the positive x -direction. Then the total flux **out** of the section $a < x < b$ is

$$A(b)q(b, t) - A(a)q(a, t)$$

(E.g. imagine people moving along a corridor point a to b of width A : ρ measures people density (0.1 people per square metre), E is total number of people in the corridor. Then q measures the flow rate of people at a point, (0.05 people per second) but it's clearly proportional to the width of the exit hence multiply by A .)

2.3.1 Conservation Law

'Mass' cannot be created or destroyed. So:

$$\text{Rate of change of 'mass' in } V = - (\text{flux of 'mass' out of } V). \quad (2.1)$$

Mathematically,

$$\frac{dE}{dt} = -[A(b)q(b, t) - A(a)q(a, t)] = - \int_a^b \frac{\partial}{\partial x} (A(x)q(x, t)) dx$$

by fundamental theorem of Calculus. Or

$$\int_a^b \frac{\partial}{\partial t} (A(x)\rho(x,t)) dx = - \int_a^b \frac{\partial}{\partial x} (A(x)q(x,t)) dx$$

since a and b are fixed. Since they are also arbitrary, must have

$$A(x) \frac{\partial}{\partial t} \rho(x,t) + \frac{\partial}{\partial x} [A(x)q(x,t)] = 0 \quad (2.2)$$

The A 's cancel when it is assumed A is constant.

Heat source

We can also assume that at each point x there is a generation of heat energy (or "mass" creation) per unit time and we call it $Q(x,t)$. Then the energy is created inside the domain and the equation becomes

$$\frac{dE}{dt} = - \int_a^b \frac{\partial}{\partial x} (A(x)q(x,t)) dx + \int_a^b Q(x,t)A(x) dx,$$

leading to the following relation

$$A(x) \frac{\partial}{\partial t} \rho(x,t) + \frac{\partial}{\partial x} [A(x)q(x,t)] = Q(x,t)A(x) \quad (2.3)$$

2.3.2 The diffusion/heat equation

In order to obtain a closed equation we need to relate $\rho(x,t)$ and $q(x,t)$.

We assume $A(x) = \text{const}$ and consider E to be the thermal energy. In this case $\rho(x,t) = \rho_0 c_v T(x,t)$ where T is temperature, ρ_0, c_v are density and specific heat capacity.

Relate flux to temperature by "Fourier's Heat law":

$$q(x,t) = -kT_x(x,t)$$

I.e. flux is proportional to gradient of temperature and "heat" from high values to low values, k is called thermal diffusivity.

Then we from the conservation law we obtain:

$$\frac{\partial}{\partial t} T(x,t) = \frac{\partial}{\partial x} \left(\frac{k}{\rho_0 c_v} \frac{\partial}{\partial x} T(x,t) \right) + \frac{1}{\rho_0 c_v} Q(x,t)$$

Defn: Let $D = k/(\rho_0 c_v) > 0$ be called the **diffusivity** or **diffusion constant**.

Then we have

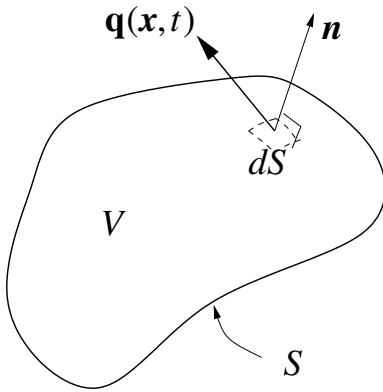
$$T_t = \partial_x (D \partial_x T) + f(x,t)$$

and if D is constant (nearly always)

$$T_t = D T_{xx} + f(x,t)$$

For chemical concentration, same principles apply: we need to use Fick's law, relating concentration and flux as $q(x,t) \sim \rho_x(x,t)$, D still appears but defined in terms of other physical parameters.

2.4 The 2D/3D heat equation



Generalise. Let $\rho(x, y, z, t) = \rho(\mathbf{x}, t)$ be 'density' of 'stuff'. Total amount of stuff in volume V , is

$$\int_V \rho(\mathbf{x}, t) dV$$

Need a **flux vector**, $\mathbf{q}(\mathbf{x}, t)$ to represent the rate of transport of 'stuff' at $\mathbf{x}$.

Flux through an elemental area dS on surface S enclosing V is

$$\mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n} dS$$

Need to take component normal to dS , (i.e. dot prod with normal $\mathbf{n}$). If $\mathbf{q} \cdot \mathbf{n} > 0$, stuff is leaving V , and vice versa.

2.4.1 Conservation Law

Apply principle (2.1):

$$\frac{d}{dt} \int_V \rho(\mathbf{x}, t) dV = - \int_S \mathbf{q} \cdot \mathbf{n} dS$$

Now using the **Divergence Theorem**, and since V is fixed,

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_V \nabla \cdot \mathbf{q} dV$$

Thus must hold for all volumes V , so

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q} = 0$$

Finally, implement the generalised version of **Fourier's law**:

$$\mathbf{q} = -D \nabla \rho$$

(stuff will flow in direction of steepest gradient downhill), then

$$\rho_t = \nabla \cdot (D \nabla \rho)$$

If D constant, then using $\nabla \cdot \nabla = \partial_{xx} + \partial_{yy} + \partial_{zz} \equiv \nabla^2$, we have

$$\rho_t = D \nabla^2 \rho$$

Note: We can add internal sources as before.

2.5 Laplace's Equation

Is the **steady state** heat/diffusion equation. I.e. no variation with time, t and so given by

$$\nabla^2 u = 0$$

where:

- In 2D: $\nabla^2 u \equiv u_{xx} + u_{yy} = 0$.
- In 3D: $\nabla^2 u \equiv u_{xx} + u_{yy} + u_{zz} = 0$.

Functions u satisfying Laplace's equation are said to be **Harmonic**.

(Also used for **potential flow** in fluids & electrostatic potentials, soap films etc...)

Chapter 3

PDE's on finite intervals in 1D. Separation of variables method.

In this chapter we will be solving diffusion/heat equation and wave equation on a finite interval. The main point of the chapter is to illustrate a powerful technique for solving linear (and some nonlinear) PDEs – **method of separation of variables**.

3.1 Diffusion/heat equation

The simplest form of diffusion/heat equation on a finite interval (a, b) is

$$u_t = Du_{xx}, \quad t > 0, x \in (a, b). \quad (3.1)$$

Here D is diffusion constant. The **dimensions** of D can be found from looking at the equation (3.1) and taking dimensions, $[.]$, of both sides

$$\frac{[u]}{[t]} = [D] \frac{[u]}{[x^2]}$$

The dimensions of t are time T , and the dimensions of x is length L . So dimensions of D are

$$[D] = L^2/T$$

(e.g. m^2/s , mm^2/hour).

E.g. (i)

When u represents temperature (in Kelvin), D is different for different media. Larger in metals – conduct heat well. Smaller in insulating materials. In standard SI units (metres, seconds):

medium	$D(\text{m}^2/\text{s})$
copper	11.2×10^{-5}
air	2.2×10^{-5}
glass	0.034×10^{-5}
water	0.015×10^{-5}

E.g. (ii)

If u measures alcohol concentration in water, $D \approx 10^{-9} \text{ m}^2/\text{s}$.

3.1.1 How diffusion constant D affects the equation

Looking at the diffusion/heat equation

$$u_t = Du_{xx}, \quad \text{where } D > 0$$

we observe that u_{xx} is the rate of change of slope of $u(t, x)$ at some fixed time t and u_t is a rate of change of $u(t, x)$ at some fixed point x .

So

$$u_t \text{ is } \begin{cases} > 0 & \text{for graph of } u \text{ convex} \\ = 0 & \text{for graph of } u \text{ straight} \\ < 0 & \text{for graph of } u \text{ concave} \end{cases}$$

Therefore at all points x where

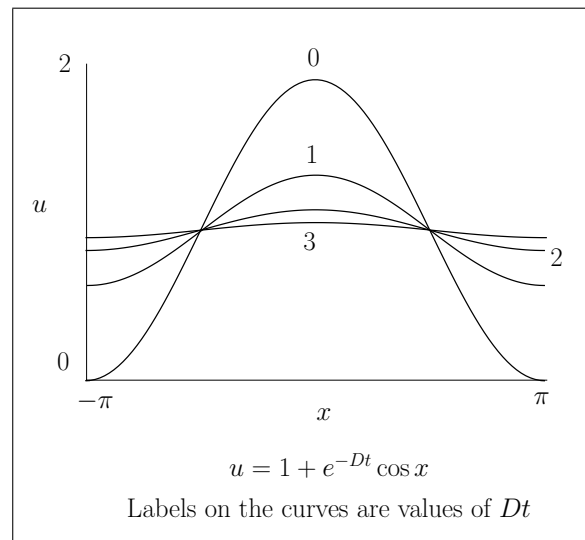
$u_{xx} > 0$ we have $u(x, t)$ is increasing as time t is increasing and at all points x where $u_{xx} < 0$ we have $u(x, t)$ is decreasing as time t is increasing.

Hence u changes to smooth out bumps.

E.g. $u(x, t) = 1 + e^{-Dt} \cos x$
clearly satisfies $u_t = Du_{xx}$.

Consider $D = 1$ first:

Graph on $-\pi < x < \pi$
shows initial central concentration spreading out and becoming uniform as t increases.



Note how u increases where $u_{xx} > 0$, and decreases where $u_{xx} < 0$.

Note: Changing D affects the rate: larger D means faster smoothing and vice versa.

3.1.2 Initial and Boundary Conditions

We observe that diffusion/heat equation (3.1) contains u_t and time interval is $t > 0$. From the theory of ODEs we recall a general principle that when we are solving **first order** ODE (initial value problem) we need **one** condition on the unknown function (initial condition) to determine it. Since we want to predict the distribution of concentration/temperature $u(x, t)$ for all $t > 0$ and u has only one derivative in t then at every x we need to prescribe one initial condition for $u(x, t)$ at time $t = 0$, i.e.

$$u(x, 0) = \phi(x).$$

On the other hand, since equation (3.1) contains u_{xx} and $x \in (a, b)$ we need to prescribe **one boundary condition** at the end points a and b at each time $t > 0$. (It is consistent with the general principle of ODEs – to solve **second order** boundary value problem one needs **two boundary conditions** (one at each end point)). These boundary conditions are determined by a physical modelling and might contain $u(\cdot, t)$ and $u_x(\cdot, t)$. The most common types of boundary conditions are

- **Dirichlet:** $u(a, t) = h(t)$, $u(b, t) = g(t)$. These boundary conditions correspond to **prescribed** temperature/concentration u at the end points of the interval (a, b) .
- **Neumann:** $u_x(a, t) = h(t)$, $u_x(b, t) = g(t)$. In this case we prescribe a flux of u rather than u itself at the end points of the interval (a, b) . In particular case of $h(t) = 0$ (or $g(t) = 0$) the end point a (or b) is **insulated**.
- **Mixed:** $u_x(a, t) = h(t)$, $u(b, t) = g(t)$ or $u(a, t) = h(t)$, $u_x(b, t) = g(t)$.
- **Periodic:** It is more convenient to consider the problem with periodic boundary conditions on the symmetric interval $(-a, a)$. Therefore boundary conditions in this case are

$$u(-a, t) = u(a, t), \quad u_x(-a, t) = u_x(a, t).$$

There is a generalization of mixed boundary condition sometimes called **Robin** boundary condition

$$\alpha u(0, t) + u_x(0, t) = h(t), \quad \beta u(a, t) + u_x(a, t) = g(t).$$

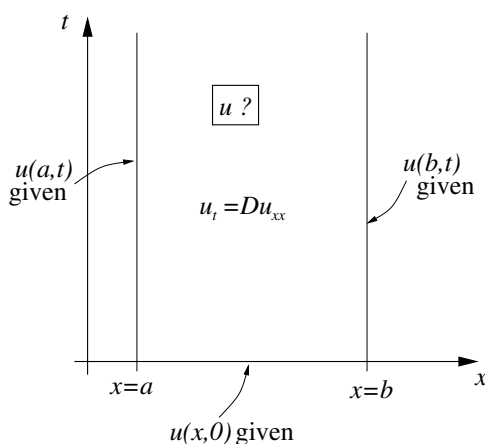
We will not be considering it here but the methods used below work for it as well.

Key E.g.:

Consider a rod of conducting metal in $a < x < b$ with **prescribed** temperatures at the two ends, and non-uniform initial temperature

Defn: Mathematically we have an **Initial and Boundary-value Problem**

- Solution evolves according to **PDE** $u_t = Du_{xx}$, for $t > 0, a < x < b$.
- $u(a, t), u(b, t)$ given for all $t > 0$ (**B.C.'s**),
- At $t = 0$, $u(x, 0)$ for $a < x < b$ is given (**Initial Condition**),
- Want to find $u(x, t)$ for $t > 0, a < x < b$.



3.2 Solution by Separation of Variables

Key technique of this chapter.

Our goal is to solve the following problem

$$u_t = Du_{xx} + f(x, t), \quad x \in (0, a), t > 0 \quad (3.2)$$

$$u(x, 0) = \phi(x), \quad (3.3)$$

and u satisfies one of the above boundary conditions. (Note that by considering interval $(0, a)$ instead of (a, b) we don't lose generality as we can always shift our solution.) To illustrate the method we solve the heat equation with Dirichlet boundary conditions

$$u(0, t) = h(t), \quad u(a, t) = g(t) \quad (3.4)$$

3.2.1 The Basic Idea

We first consider a problem (3.2)-(3.4) when $f(x, t) = 0$, $h(t) = 0$, $g(t) = 0$ and use the method of separation of variables to obtain solution. We are solving the following problem

$$u_t = Du_{xx}, \quad x \in (0, a), t > 0 \quad (3.5)$$

$$u(x, 0) = \phi(x), \quad (3.6)$$

$$u(0, t) = 0, \quad u(a, t) = 0, \text{ for } t > 0. \quad (3.7)$$

The idea behind the method of separation of variables is to look for a solution in the following form

$$u(x, t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ will be determined. Let's assume that we have a solution $u(x, t)$ in the above form. Plugging it into the equation we obtain

$$X(x)T'(t) = DX''(x)T(t),$$

Dividing both parts by $DT(t)X(x)$ we arrive to the following equality

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)}.$$

Since the left side is a function of t only and the right side is the function of x only the above equality is possible if and only if

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)} = k$$

for some constant k . Therefore if $u(x, t) = X(x)T(t)$ is a solution of the heat equation then

$$X''(x) = kX(x), \quad \text{and} \quad T'(t) = kDT(t).$$

Moreover, from the boundary conditions $u(0, t) = u(a, t) = 0$ we know that $X(0)T(t) = 0$ and $X(a)T(t) = 0$ for all $t > 0$. This implies $X(0) = 0$ and $X(a) = 0$.

We can easily solve the problem for $T(t)$ to obtain

$$T(t) = Ce^{-kDt},$$

where we don't know constants C and k .

3.2.2 Problem for $X(x)$ – the role of boundary conditions

The problem for $X(x)$ is called an eigenvalue problem

$$X''(x) = kX(x) \text{ on } (0, a), \quad X(0) = 0, \quad X(a) = 0. \quad (3.8)$$

In order to solve it we have to find all constants k and all nontrivial solutions $X(x)$, corresponding to these k . It is not difficult to see that $k < 0$. Indeed, if we multiply both parts of the equation (3.8) by $X(x)$ and integrate by parts we obtain

$$\int_0^a |X'(x)|^2 + k|X(x)|^2 dx = 0.$$

Obviously, if $k \geq 0$ then $X(x) = 0$ on $(0, a)$. Since we are interested in non-trivial solutions we have to consider only $k = -\lambda^2 < 0$. In that case the general solution of (3.8) is

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x)$$

and constants A, B, λ will be found from the boundary conditions $X(0) = X(a) = 0$. Let's consider $X(0) = 0$: obviously $X(0) = A$ and therefore $A = 0$. Taking this into account and using $X(a) = 0$ we obtain

$$B \sin(\lambda a) = 0.$$

It's clear that $B \neq 0$ since if $B = 0$ the solution $X(x)$ is zero everywhere and we are interested only in non-zero solutions. Therefore $\lambda = \frac{\pi n}{a}$ for all $n \in \mathbb{Z} \setminus \{0\}$. The solution of the eigenvalue problem is

$$k = \frac{\pi^2 n^2}{a^2}, \quad n \in \mathbb{N}^+ = \{1, 2, 3, \dots\}, \quad X(x) = B \sin\left(\frac{\pi n}{a}x\right).$$

Combining our information about $k, X(x)$ and $T(t)$ we arrive to the conclusion that we have infinitely many solutions of the heat equation (3.5), satisfying the boundary conditions (3.7). Indeed, for all $n \in \mathbb{N}^+$ a function

$$u_n(x, t) = e^{-\frac{\pi^2 n^2}{a^2}Dt} \sin\left(\frac{\pi n}{a}x\right)$$

solves (3.5) and (3.7).

Principle of superposition

Each solution above satisfies the linear homogeneous PDE and the linear homogeneous boundary conditions. Therefore we can use superposition and sum arbitrary numbers of each solution

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2 \pi^2 D t}{a^2}} \sin\left(\frac{n \pi x}{a}\right) \quad (3.9)$$

Here a_n are arbitrary constants and we call it the **general solution**. This general solution satisfies PDE and boundary conditions but it does not necessarily satisfy the initial condition. We still have to find the solution that satisfies (3.6) and that means finding a_n -s using initial data.

3.2.3 The Initial Condition

The particular solution is determined by initial temperature distribution

$$u(x, 0) = \phi(x) \text{ for } 0 < x < a$$

where ϕ is a **given** function of x .

From (3.9) we have

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) = \phi(x) \text{ for } 0 < x < a \quad (3.10)$$

This essentially determines a_n . **Informally**, we proceed as follows: multiply both parts of (3.10) by a function $\sin\left(\frac{m\pi x}{a}\right)$, $m \in \mathbb{N}^+$ and integrate between 0 and a

$$\sum_{n=1}^{\infty} a_n \int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \int_0^a \phi(x) \sin\left(\frac{m\pi x}{a}\right) dx$$

It is straightforward to compute

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \begin{cases} a/2 & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$$

Therefore we have

$$\frac{a}{2} a_m = \int_0^a \phi(x) \sin\left(\frac{m\pi x}{a}\right) dx$$

and for all $n \in \mathbb{N}^+$

$$a_n = \frac{2}{a} \int_0^a \phi(x) \sin\left(\frac{\pi n}{a} x\right) dx.$$

Finally, the solution to the problem (3.5), (3.6), (3.7) is

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{\pi n}{a}\right)^2 D t} \sin\left(\frac{\pi n}{a} x\right),$$

where

$$a_n = \frac{2}{a} \int_0^a \phi(x) \sin\left(\frac{\pi n}{a} x\right) dx.$$

In order for this procedure to make sense we have to justify the representation (3.10) for our initial data $\phi(x)$. This is done through the **Fourier series**.

3.2.4 Fourier Series

Let us consider a function $f(x)$ on an interval $(-a, a)$ and define a Fourier series as

$$\text{Fourier series} = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{a}\right). \quad (3.11)$$

We want to understand how to represent $f(x)$ in terms of a **Fourier series**.

Informally equating $f(x)$ and FS , we can find the coefficients using orthogonality relation as in the previous section to obtain

$$a_0 = \frac{1}{2a} \int_{-a}^a f(x) dx, \quad a_n = \frac{1}{a} \int_{-a}^a f(x) \cos\left(\frac{\pi nx}{a}\right) dx, \quad b_n = \frac{1}{a} \int_{-a}^a f(x) \sin\left(\frac{\pi nx}{a}\right) dx. \quad (3.12)$$

Defn: We define (3.11) with coefficients (3.12) to be the **Fourier series** of $f(x)$.

We have the following result that is supplied without a proof.

Theorem. If $f(x)$ is a piecewise smooth function on the interval $[-a, a]$ then the Fourier series of $f(x)$ converges pointwise

1. to $f(x)$, where $f(x)$ is continuous at $x \in (-a, a)$;
2. to $\frac{1}{2}(f(x+) + f(x-))$, where $f(x)$ has a jump at $x \in (-a, a)$;

Moreover, at points $-a$ and a the Fourier series converges to $\frac{1}{2}(f(-a) + f(a))$.

We want to note several points

- the interval of definition of $f(x)$ is symmetric $[-a, a]$;
- if $f(x)$ is an odd function on $[-a, a]$ then all coefficients $a_n = 0, n \in \mathbb{N}$;
- if $f(x)$ is an even function on $[-a, a]$ then all coefficients $b_n = 0, n \in \mathbb{N}$;
- if $f(x)$ is defined on the interval $[0, a]$ we can always **extend** it to the interval $[-a, a]$ using either **odd** or **even** extensions. In particular, if $f(0) = 0$ then odd extension of $f(x)$ is continuous and if $f'(0) = 0$ then even extension of $f(x)$ is smooth.

Examples:

e.g. 1: Let $a = 1, \phi(x) = x(1 - x)$. Using odd extension and sin series we can calculate

$$\begin{aligned} a_n &= 2 \int_0^1 x(1 - x) \sin(n\pi x) dx \\ &= 2 \left[\frac{x(1 - x) \cos(n\pi x)}{-n\pi} \right]_0^1 + \frac{2}{n\pi} \int_0^1 (1 - 2x) \cos(n\pi x) dx \\ &= \frac{2}{n\pi} \left[\frac{(1 - 2x) \sin(n\pi x)}{n\pi} \right]_0^1 - \frac{2}{n^2\pi^2} \int_0^1 (-2) \sin(n\pi x) dx \\ &= \frac{4}{n^2\pi^2} \left[\frac{\cos(n\pi x)}{-n\pi} \right]_0^1 \\ &= \frac{4(1 - \cos(n\pi))}{n^3\pi^3} \end{aligned}$$

and note that $\cos(n\pi) = (-1)^n$.

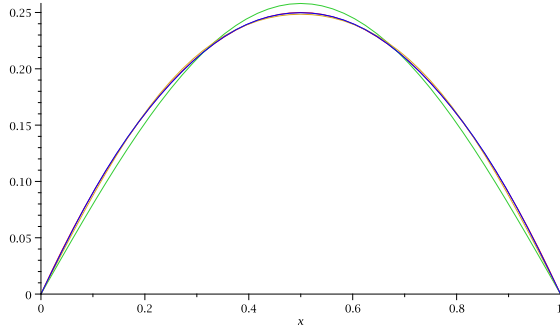
Now

$$s_N(x) = \sum_{n=1}^N \frac{4(1 - (-1)^n) \sin(n\pi x)}{n^3\pi^3}$$

or, since $1 - (-1)^n = 0$ when n even and is 2 when n odd, let $n = 2r - 1, r = 1, 2, 3, \dots$ so that

$$s_{2N-1}(x) = \sum_{r=1}^N \frac{8 \sin((2r-1)\pi x)}{(2r-1)^3 \pi^3}$$

Plot $s_3(x), s_5(x)$ and $s_9(x)$ alongside $\phi(x)$. Better fit for larger N .



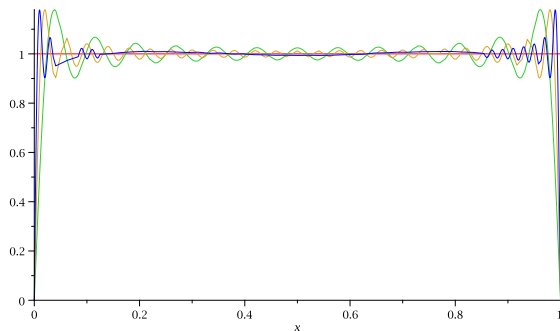
e.g. 2: Let $a = 1, \phi(x) = 1$. Again, using odd extension and sin series we have

$$a_n = 2 \int_0^1 \sin(n\pi x) dx = 2 \left[\frac{\cos(n\pi x)}{-n\pi} \right]_0^1 = \frac{2(1 - \cos(n\pi))}{n\pi} = \frac{2(1 - (-1)^n)}{n\pi}$$

As above, make substitution $r = 2n - 1$ to pick out odd terms only

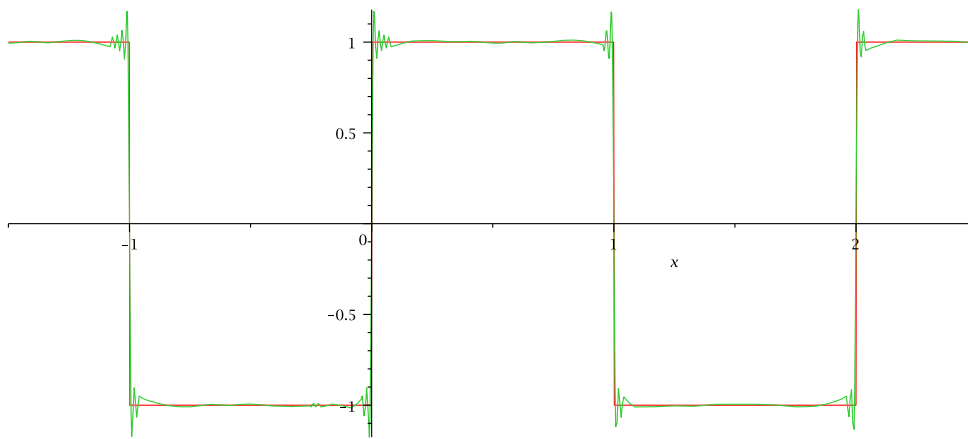
$$s_{2N-1}(x) = \sum_{r=1}^N \frac{4 \sin((2r-1)\pi x)}{(2r-1)\pi}$$

Plot $s_{25}(x), s_{49}(x)$ and $s_{99}(x)$ alongside $\phi(x)$.



Note that convergence is much worse in e.g.2 than in e.g.1: Of course, $s_N(0) = s_N(1) = 0$ for all N and this contradicts the definition of $\phi(x)$.

This is because the Fourier sine series representation, and hence $s_N(x)$ exists for $x \in \mathbb{R}$, not just in $0 < x < 1$, where $\phi(x)$ is defined. Thus $s_N(x)$, and is an **odd**, periodic function with periodicity in x of 2. The odd periodic extension of the original functions are



Fourier Series representation will always converge to the mean value of a discontinuity. The partial sums will exhibit wild oscillations and slowly convergence near these points. This is known as the **Gibbs phenomenon**.

3.2.5 Examples of heat equation with particular initial conditions

In our example of the diffn. equation with $u(0, t) = u(a, t) = 0$ we had a general solution

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2 \pi^2 D t}{a^2}} \sin\left(\frac{n \pi x}{a}\right)$$

with

$$a_n = \frac{2}{a} \int_0^a \phi(x) \sin(n \pi x / a) dx$$

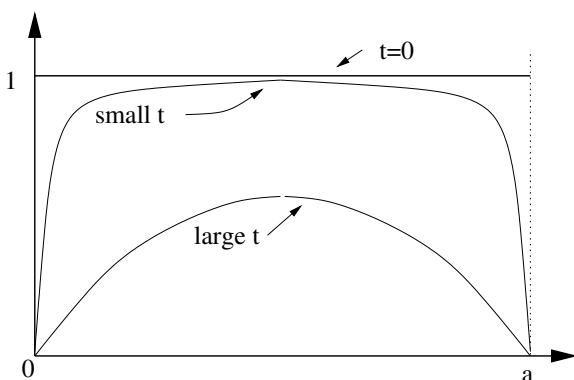
Example 1. if the I.C. is $u(x, 0) = \phi(x) = 1$ for $0 < x < a$ then

$$a_n = \frac{2}{a} \int_0^a \sin(n \pi x / a) dx = -\left(\frac{2}{n \pi}\right) [\cos(n \pi) - 1] = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n \pi} & \text{for } n \text{ odd} \end{cases}$$

So

$$u(x, t) = \left(\frac{4}{\pi}\right) \sum_{\text{odd } n} \frac{1}{n} e^{-\frac{n^2 \pi^2 D t}{a^2}} \sin\left(\frac{n \pi x}{a}\right) \text{ for } 0 < x < a$$

Let $r = 2n + 1$ to make sum explicit.



Notes:

- Instantly drops to 0 at ends.
- If $t > a^2 / \pi^2 D$ then $e^{-n^2 \pi^2 D t / a^2} \ll e^{-\pi^2 D t / a^2}$ for $n \geq 3$. I.e. first term dominates; profile in x is approx. $\sin(\pi x / a)$.
- This is the **SMOOTHING** effect of diffusion.

Example 2. Let $a = 1$ and $\phi(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < x < 1 \end{cases}$

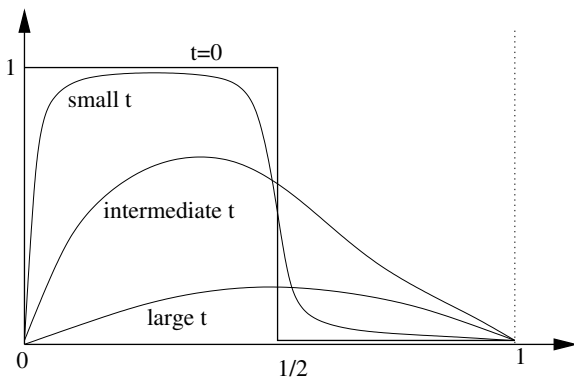
get

$$a_n = 2 \int_0^{1/2} \sin(n\pi x) dx = 2 \left[\frac{-\cos(n\pi x)}{n\pi} \right]_0^{1/2} = \dots$$

so

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{1 - \cos(\frac{1}{2}n\pi)}{n} \right) e^{-n^2 \pi^2 D t} \sin n\pi x$$

$1 - \cos(\frac{1}{2}n\pi) = 2 \sin^2(\frac{1}{4}n\pi)$, takes values 1, 2, 1, 0, 1, 2, 1, 0, ... as $n = 1, 2, \dots$



Notes:

- initially a step, immediately becomes smooth; gets more symmetric and sinusoidal as it decays.
- When $Dt > 1 / \pi^2$, first term dominates.

3.2.6 Overview of Separation of Variables Method

To solve PDE for $u(x, t)$ for $0 \leq x \leq a$ with homogeneous boundary conditions, zero right hand side and given initial values:

1. Separate variables $u(x, t) = X(x)T(t)$, giving ODEs for $X(x)$ and $T(t)$ involving an unknown separation constant k
2. Solve for $X(x)$ using BCs to determine values for separation constant k , and eigenfunctions $X(x)$.

3. Solve equation for $T(t)$ and get an infinite set of solutions $u_n(t, x)$
4. Take $u(x, t)$ to be infinite series of $u_n(x, t)$ with arbitrary coefficients a_n .
5. Find a_n by matching $u(x, 0)$ to the given initial condition using Fourier series.

3.2.7 Heat equation with $f(x, t) \neq 0$ and homogeneous boundary conditions

Now as we understand the basic idea of separation of variables method we continue towards our goal of solving the problem (3.1), (3.3), (3.4). We take another step and consider the following problem

$$u_t = Du_{xx} + f(x, t), \quad \text{for } x \in (0, a), t > 0 \quad (3.13)$$

$$u(x, 0) = \phi(x), \quad (3.14)$$

$$u(0, t) = 0, \quad u(a, t) = 0, \quad \text{for } t > 0. \quad (3.15)$$

From the above analysis we know that a Fourier series with basis functions $\sin\left(\frac{\pi n}{a}x\right)$, $n \in \mathbb{N}^+$ is consistent with the boundary conditions. We also know that for a fixed t any function $u(x, t)$ and $f(x, t)$ can be expanded in a Fourier series as follows

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{\pi n}{a}x\right),$$

where

$$f_n(t) = \frac{2}{a} \int_0^a f(x, t) \sin\left(\frac{\pi n}{a}x\right) dx \quad \text{for } n \in \mathbb{N}^+,$$

and

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{\pi n}{a}x\right),$$

where $u_n(t)$ will be determined later using equation.

Plugging these representations for $u(x, t)$ and $f(x, t)$ in the (3.13) we obtain

$$\sum_{n=1}^{\infty} \left(u'_n(t) + D \frac{\pi^2 n^2}{a^2} u_n(t) - f_n(t) \right) \sin\left(\frac{\pi n}{a}x\right) = 0.$$

Multiplying both parts of equality by $\sin\left(\frac{\pi m}{a}x\right)$ and integrating over $(0, a)$ we have

$$u'_m(t) + D \frac{\pi^2 m^2}{a^2} u_m(t) = f_m(t).$$

Using initial data we clearly have

$$u(x, 0) = \sum_{n=1}^{\infty} u_n(0) \sin\left(\frac{\pi n}{a}x\right) = \phi(x),$$

and therefore

$$u_m(0) = \frac{2}{a} \int_0^a \phi(x) \sin\left(\frac{\pi m}{a}x\right) dx \quad \text{for } m \in \mathbb{N}^+.$$

Now for each $m \in \mathbb{N}^+$ we have the following ODE

$$u'_m(t) + D \frac{\pi^2 m^2}{a^2} u_m(t) = f_m(t), \quad u_m(0) = \frac{2}{a} \int_0^a \phi(x) \sin\left(\frac{\pi m}{a} x\right) dx.$$

It is clear that we can solve it to obtain

$$u_m(t) = u_m(0) e^{-D \frac{\pi^2 m^2}{a^2} t} + e^{-D \frac{\pi^2 m^2}{a^2} t} \int_0^t e^{D \frac{\pi^2 m^2}{a^2} s} f_m(s) ds. \quad (3.16)$$

Since we know $u_m(0)$ and $f_m(t)$ we obtain $u_m(t)$ and therefore we have a solution of the problem (3.13)-(3.15) as

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{\pi n}{a} x\right),$$

where u_n are given in (3.16).

Here we used the idea of representing the solution $u(x, t)$ in terms of a Fourier series with basis functions consistent with the boundary conditions. This basis was found before while solving fully homogeneous problem.

3.2.8 Inhomogeneous boundary conditions

In this subsection we want to understand how to solve the following problem

$$u_t = Du_{xx}, \quad \text{for } x \in (0, a), t > 0 \quad (3.17)$$

$$u(x, 0) = 0, \quad (3.18)$$

$$u(0, t) = h(t), \quad u(a, t) = g(t), \text{ for } t > 0. \quad (3.19)$$

The idea is to find an explicit function $v(x, t)$ that will satisfy the boundary condition (3.19) and then look at what equation will be satisfied by a difference $u(x, t) - v(x, t)$. The construction of an auxiliary function is simple:

$$v(x, t) = \frac{g(t) - h(t)}{a} x + h(t)$$

satisfies $v(0, t) = h(t)$ and $v(a, t) = g(t)$. Here we took a linear interpolation between two boundary conditions but **any** other construction that makes $v(x, t)$ satisfy (3.19) is also good.

Now we look at the difference $w(x, t) = u(x, t) - v(x, t)$. It is clear that w satisfies the following problem

$$w_t = Dw_{xx} - v_t(x, t), \quad \text{for } x \in (0, a), t > 0 \quad (3.20)$$

$$w(x, 0) = -v(x, 0), \quad (3.21)$$

$$w(0, t) = 0, \quad w(a, t) = 0, \text{ for } t > 0. \quad (3.22)$$

Using results from the previous subsection we can find $w(x, t)$ and therefore we can find

$$u(x, t) = w(x, t) + v(x, t),$$

since we know both w and v .

Note that the same idea works for Neumann and Mixed boundary conditions. We just need to construct an appropriate auxiliary function $v(x, t)$ satisfying boundary conditions.

3.2.9 General problem

Now we are ready to solve the following general problem

$$u_t = Du_{xx} + f(x, t), \quad \text{for } x \in (0, a), t > 0 \quad (3.23)$$

$$u(x, 0) = \phi(x), \quad (3.24)$$

$$u(0, t) = h(t), \quad u(a, t) = g(t), \text{ for } t > 0. \quad (3.25)$$

It is straightforward to check that if $v(x, t)$ satisfies

$$v_t = Dv_{xx} + f(x, t), \quad \text{for } x \in (0, a), t > 0 \quad (3.26)$$

$$v(x, 0) = \phi(x), \quad (3.27)$$

$$v(0, t) = 0, \quad v(a, t) = 0, \text{ for } t > 0, \quad (3.28)$$

and $w(x, t)$ satisfies

$$w_t = Dw_{xx}, \quad \text{for } x \in (0, a), t > 0 \quad (3.29)$$

$$w(x, 0) = 0, \quad (3.30)$$

$$w(0, t) = h(t), \quad w(a, t) = g(t), \text{ for } t > 0, \quad (3.31)$$

then $u(x, t) = v(x, t) + w(x, t)$ solves (3.23), (3.24), (3.25). Since we already know from the previous subsections how to solve the problems for $v(x, t)$ and $w(x, t)$ we are done.

We solved the diffusion/heat equation with Dirichlet boundary conditions. However, the same method of separation of variables works for all types of boundary conditions. As another example we consider homogeneous Neumann boundary conditions.

3.2.10 Homogeneous heat equation with Neumann boundary conditions

As an example we are solving the following problem

$$u_t = Du_{xx}, \quad \text{for } x \in (0, a), t > 0 \quad (3.32)$$

$$u(x, 0) = \phi(x), \quad (3.33)$$

$$u_x(0, t) = 0, \quad u_x(a, t) = 0, \text{ for } t > 0. \quad (3.34)$$

We use the same method of separation of variables and obtain the following eigenvalue problem for $X(x)$

$$X''(x) = kX(x), \text{ on } (0, a), \quad X'(0) = 0, \quad X'(a) = 0. \quad (3.35)$$

In this case we can show by the same methods as before that $k = -\lambda^2 \leq 0$ and therefore the general solution is

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x).$$

Using $X'(0) = 0$ we obtain $B = 0$ and using $X'(a) = 0$ we obtain $A \sin(\lambda a) = 0$. Therefore $\lambda = \frac{\pi n}{a}$ for all $n \in \mathbb{Z}$ and the solution of eigenvalue problem (3.35) is

$$k = -\frac{\pi^2 n^2}{a^2}, \quad n \in \mathbb{N} = \{0, 1, 2, 3, \dots\}, \quad X(x) = A \cos\left(\frac{\pi n}{a}x\right).$$

Note that here we allow for $n = 0$ in which case $X(x) = A$ is a solution. As before we obtain the solution

$$u(x, t) = \sum_{n=0}^{\infty} a_n e^{-\frac{\pi^2 n^2}{a^2} D t} \cos\left(\frac{\pi n}{a} x\right),$$

where we need to find a_n -s from initial data.

By the same arguments as before we have to find a_n from the following equality

$$\phi(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n}{a} x\right).$$

Note that here we need to match ϕ with a **cosine Fourier series**. Multiplying both parts by $\cos\left(\frac{\pi m}{a} x\right)$, $m \in \mathbb{N}$ and integrating over $(0, a)$ we obtain

$$a_m = \frac{2}{a} \int_0^a \phi(x) \cos\left(\frac{\pi m}{a} x\right) dx \quad \text{for } n > 0, \quad a_0 = \frac{1}{a} \int_0^a \phi(x) dx.$$

Therefore the solution is

$$u(x, t) = \sum_{n=0}^{\infty} a_n e^{-\frac{\pi^2 n^2}{a^2} D t} \cos\left(\frac{\pi n}{a} x\right),$$

where

$$a_n = \frac{2}{a} \int_0^a \phi(x) \cos\left(\frac{\pi n}{a} x\right) dx \quad \text{for } n > 0, \quad a_0 = \frac{1}{a} \int_0^a \phi(x) dx.$$

Example 1. $\phi(x) = 1$ for $0 < x < a$.

Then

$$a_0 = \frac{1}{a} \int_0^a 1 \cos(0\pi x/a) dx = 1$$

whilst

$$a_n = \frac{2}{a} \int_0^a 1 \cdot \cos(n\pi x/a) dx = 2 \left[\frac{\sin(n\pi x/a)}{n\pi} \right]_0^a = 0$$

So $u(x, t) = a_0 = 1 \dots$

Why so simple ? Easy to see $u = 1$ satisfies B.C.'s and PDE.

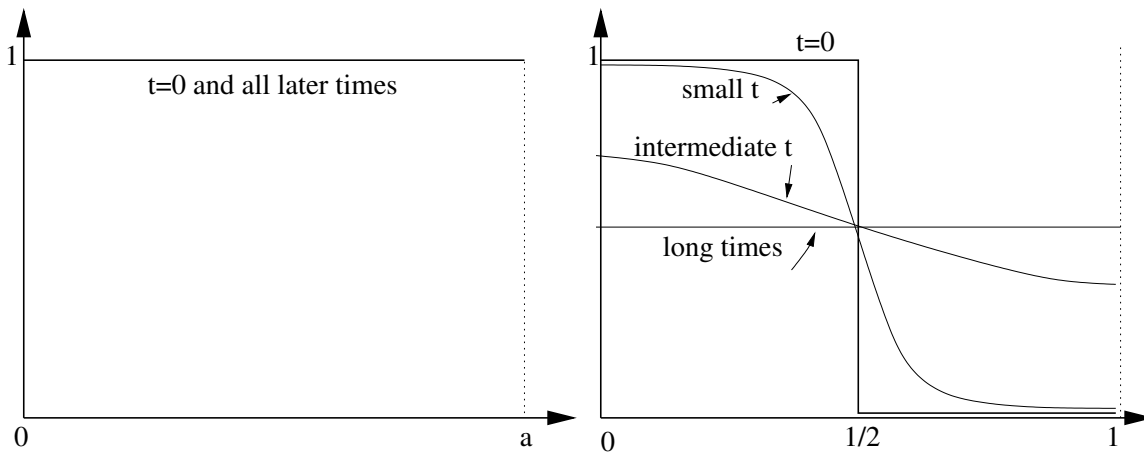
Example 2. Take $a = 1$ and $\phi(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < x < 1 \end{cases}$

$$a_0 = \int_0^1 \phi(x) dx = \frac{1}{2}, \quad a_n = 2 \int_0^{\frac{1}{2}} \cos n\pi x dx = (2/n\pi) \sin(\frac{1}{2}n\pi).$$

$$u(x, t) = \frac{1}{2} + \left(\frac{2}{\pi}\right) \sum_{n=1}^{\infty} \left(\frac{\sin(\frac{1}{2}n\pi)}{n}\right) e^{-n^2\pi^2 D t} \cos n\pi x$$

In fact

$$\sin(\frac{1}{2}n\pi) = \begin{cases} 0 & \text{for } n \text{ even} \\ (-1)^r & \text{for } n = 2r + 1 \end{cases}$$



Note: Sharp corner gets immediately smoothed out; solution evens out to its average value.

3.2.11 The Role of the Diffusivity

D appears in the solution only in the terms $e^{-n^2\pi^2Dt}$, for e.g.

Changing D equivalent to rescaling t : doubling D is the same as things happens twice as fast.

$D = 0$ means infinitely slow time-scale: all exponentials in the solution are 1, so solution is constant (obvious from PDE: $u_t \equiv 0$ when $D = 0$).

The Time-Reversed Diffusion Equation: Blowup

If $D < 0$ then same as changing sign of t which gives $-$ sign. Either way,

$$u_t = -Du_{xx}, \quad \text{now } D > 0$$

Effect is to give $e^{+n^2\pi^2Dt}$ in the solution where $D > 0$. This exponentially increasing term gets more dominant as t increases, and usually leads to divergence for some t .

If IC is not **exceptionally** smooth including at end-points, series diverges for all $t > 0$. In fact, if Fourier coefficients are not exponentially decaying in n , blowup is *immediate* since for any $t > 0$, the exponential terms increase exponentially with n so n -th term $\nrightarrow 0$

3.3 Wave equation

In this section we concentrate on the **wave equation**

$$u_{tt} = c^2 u_{xx}$$

Dimensions of c ? Take dimensions of the PDE:

$$\frac{[u]}{[t^2]} = [c^2] \frac{[u]}{[x^2]}$$

so that $[c^2] = [c]^2 = \text{length}^2/\text{time}^2$ or $[c] = LT^{-1}$. I.e. c has dimensions of speed (e.g. metres per second). It is the wave speed.

We observe that wave equation is 2nd order in t so $u(x, t)$ vibrates, not decays. Also we need to supply two initial conditions, $u(x, 0)$, $u_t(x, 0)$ instead of one in diffusion/heat equation.

Our goal is to solve the following problem

$$u_{tt} = c^2 u_{xx} + f(x, t), \quad x \in (0, a), \quad (3.36)$$

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \quad (3.37)$$

and u satisfies one of the boundary conditions.

In order to achieve this goal we proceed as with heat equation, first consider a problem when $f(x, t) = 0$, $h(t) = 0$, $g(t) = 0$ and use the method of separation of variables to obtain solution.

To illustrate the method we solve the wave equation with **mixed** and **periodic** boundary conditions. Dirichlet and Neumann are treated in the similar way and have been considered before for heat equation.

3.3.1 Homogeneous wave equation with mixed boundary conditions

Here we are solving the following problem

$$u_{tt} = c^2 u_{xx}, \quad \text{for } x \in (0, a), t > 0 \quad (3.38)$$

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \quad (3.39)$$

$$u_x(0, t) = 0, \quad u(a, t) = 0, \quad \text{for } t > 0. \quad (3.40)$$

We look for a solution in the following form

$$u(x, t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ will be determined. Let's assume that we have a solution $u(x, t)$ in the above form. Plugging it into the equation we obtain

$$X(x)T''(t) = c^2 X''(x)T(t),$$

Dividing both parts by $c^2 T(t)X(x)$ we arrive to the following equality

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left side is a function of t only and the right side is the function of x only the above equality is possible if and only if

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = k$$

for some constant k . Therefore if $u(x, t) = X(x)T(t)$ is a solution of the wave equation then

$$X''(x) = kX(x), \quad \text{and} \quad T''(t) = kc^2 T(t).$$

Moreover, from the boundary conditions (3.40) we know that $X'(0)T(t) = 0$ and $X(a)T(t) = 0$ for all $t > 0$. This implies $X'(0) = 0$ and $X(a) = 0$.

The problem for $X(x)$ is called an eigenvalue problem

$$X''(x) = kX(x) \text{ on } (0, a), \quad X'(0) = 0, \quad X(a) = 0. \quad (3.41)$$

In order to solve it we have to find all k -s and all nontrivial solutions $X(x)$, corresponding to these k -s. It is not difficult to see that $k < 0$. Indeed, if we multiply both parts of the equation (3.41) by $X(x)$ and integrate by parts we obtain

$$\int_0^a |X'(x)|^2 + k|X(x)|^2 dx = 0.$$

Obviously, if $k \geq 0$ then $X(x) = 0$ on $(0, a)$. Since we are interested in non-trivial solutions we have to consider only $k = -\lambda^2 < 0$. In that case the general solution of (3.41) is

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x)$$

and constants A, B, λ will be found from the boundary conditions $X'(0) = X(a) = 0$. Let's consider $X'(0) = 0$: obviously $X'(0) = B\lambda$ and therefore $B = 0$. Taking this into account and using $X(a) = 0$ we obtain

$$A \cos(\lambda a) = 0.$$

It's clear that $A \neq 0$ as then the solution is zero everywhere and we are interested only in non-zero solutions. Therefore $\lambda = \frac{1}{a} \left(\frac{\pi}{2} + \pi n \right)$ for all $n \in \mathbb{Z}$. The solution of the eigenvalue problem is

$$k = \frac{1}{a^2} \left(\frac{\pi}{2} + \pi n \right)^2, \quad n \in \mathbb{N} = \{0, 1, 2, 3, \dots\}, \quad X(x) = A \cos \left(\frac{1}{a} \left(\frac{\pi}{2} + \pi n \right) x \right).$$

Solving equation for $T(t)$ we obtain

$$T(t) = C \cos \left(\frac{c}{a} \left(\frac{\pi}{2} + \pi n \right) t \right) + D \sin \left(\frac{c}{a} \left(\frac{\pi}{2} + \pi n \right) t \right).$$

Combining our information about $X(x)$ and $T(t)$ we arrive to the conclusion that we have infinitely many solutions of the wave equation (3.38), satisfying the boundary conditions (3.40). Indeed, for all $n \in \mathbb{N}$ a function

$$u_n(x, t) = \left(a_n \cos \left(\frac{c}{a} \left(\frac{\pi}{2} + \pi n \right) t \right) + b_n \sin \left(\frac{c}{a} \left(\frac{\pi}{2} + \pi n \right) t \right) \right) \cos \left(\frac{1}{a} \left(\frac{\pi}{2} + \pi n \right) x \right)$$

solves (3.38) and (3.40). Therefore, any linear combination of u_n -s also solves (3.38) and (3.40) and we represent

$$u(x, t) = \sum_{n=0}^{\infty} \left(a_n \cos \left(\frac{c}{a} \left(\frac{\pi}{2} + \pi n \right) t \right) + b_n \sin \left(\frac{c}{a} \left(\frac{\pi}{2} + \pi n \right) t \right) \right) \cos \left(\frac{1}{a} \left(\frac{\pi}{2} + \pi n \right) x \right),$$

where a_n, b_n are arbitrary constants. We will find a_n, b_n as before using initial data.

3.3.2 Homogeneous wave equation with periodic boundary conditions

Here we are solving the following problem

$$u_{tt} = c^2 u_{xx}, \quad \text{for } x \in (-a, a), t > 0 \quad (3.42)$$

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \quad (3.43)$$

$$u(-a, t) = u(a, t), \quad u_x(-a, t) = u_x(a, t), \quad \text{for } t > 0. \quad (3.44)$$

We look for a solution in the following form

$$u(x, t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ will be determined. Let's assume that we have a solution $u(x, t)$ in the above form. Plugging it into the equation we obtain

$$X(x)T''(t) = c^2 X''(x)T(t),$$

Dividing both parts by $c^2 T(t)X(x)$ we arrive to the following equality

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left side is a function of t only and the right side is the function of x only the above equality is possible if and only if

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = k$$

for some constant k . Therefore if $u(x, t) = X(x)T(t)$ is a solution of the wave equation then

$$X''(x) = kX(x), \quad \text{and} \quad T''(t) = kc^2 T(t).$$

Moreover, from the boundary conditions (3.44) we know that $X(-a)T(t) = X(a)T(t)$ and $X'(-a)T(t) = X'(a)T(t)$ for all $t > 0$. This implies $X(-a) = X(a)$ and $X'(-a) = X'(a)$.

Solving equation for $X(x)$ we obtain

$$X(x) = A \cos(\sqrt{k}x) + B \sin(\sqrt{k}x).$$

Using boundary conditions we have

$$A \cos(\sqrt{k}a) - B \sin(\sqrt{k}a) = A \cos(\sqrt{k}a) + B \sin(\sqrt{k}a);$$

$$\sqrt{k}(A \sin(\sqrt{k}a) + B \cos(\sqrt{k}a)) = \sqrt{k}(-A \sin(\sqrt{k}a) + B \cos(\sqrt{k}a)).$$

Solving these equations we obtain

$$\sin(\sqrt{k}a) = 0.$$

Therefore $\sqrt{k} = \frac{\pi n}{a}$ for all $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and

$$X_n(x) = A_n \cos\left(\frac{\pi n}{a}x\right) + B_n \sin\left(\frac{\pi n}{a}x\right).$$

Solving equation for $T(t)$ we obtain

$$T_n(t) = C_n \cos\left(\frac{\pi n c}{a}t\right) + D_n \sin\left(\frac{\pi n c}{a}t\right).$$

Now we can find the general solution $u(x, t)$ to be

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{\pi n c}{a}t\right) + D_n \sin\left(\frac{\pi n c}{a}t\right) \right) \left(A_n \cos\left(\frac{\pi n}{a}x\right) + B_n \sin\left(\frac{\pi n}{a}x\right) \right).$$

Using initial data we can recover a_0, A_n, B_n, C_n, D_n as before.

General problem

In order to solve the general problem (3.36), (3.37) with inhomogeneous boundary conditions we can follow exactly the same ideas as in sections 3.2.8 and 3.2.9.

Chapter 4

1D PDE's on infinite Domains: Fundamental Solution

PDE's on **infinite domains** needs a new technique. In the next two chapters we consider two general methods of solving heat and wave equations on $\mathbb{R}$, namely, method of fundamental solutions (it is also sometimes called method of Green's functions) and Fourier transforms method. We will also construct a solution on the semi-infinite domain $\mathbb{R}_+$ based on the knowledge of a solution on $\mathbb{R}$.

Boundary conditions

When we are on the whole $\mathbb{R}$ we cannot prescribe boundary conditions. Usually we assume some information about behaviour of the solution at infinity, for instance, that $u_x(x, t)$ or/and $u(x, t)$ vanish as $x \rightarrow \pm\infty$. We will spell out these assumptions for particular problems.

For semi-infinite domain $\mathbb{R}_+$ we prescribe Dirichlet or Neumann boundary conditions only at one end $x = 0$ and treat infinity as described above.

4.1 Heat equation on $\mathbb{R}$. Method of fundamental solution.

Our goal is to solve the following problem

$$u_t = Du_{xx} + f(x, t), \quad x \in \mathbb{R}, t > 0, \quad (4.1)$$

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}. \quad (4.2)$$

We would like to look for a solution that is integrable and decays at infinity together with its x -derivatives, i.e.

$$\int_{\mathbb{R}} |u(x, t)| dx < \infty, \quad u_x(x, t) \rightarrow 0 \text{ for } x \rightarrow \pm\infty. \quad (4.3)$$

We also want to make reasonable assumptions about behaviour of $f(x, t)$ and $\phi(x)$ at infinity:

$$\int_{\mathbb{R}} |\phi(x)| dx < \infty, \quad \int_{\mathbb{R}} |f(x, t)| dx < \infty \quad \text{for all } t > 0.$$

4.1.1 Similarity solutions

Firstly, we will try to solve homogeneous equation without any initial data

$$u_t = Du_{xx}, \quad x \in \mathbb{R}, t > 0$$

using *similarity solutions*. It is clear that rescaling of $\mathbb{R}$ by a constant will always give the same domain $\mathbb{R}$. Therefore if we change space variable $z = ax$ and change time variable $s = a^2t$ (it's called **dilation**), we will obtain exactly the same domains for x and t . Moreover we see that if $u(x, t)$ is a solution of the heat equation then

$$v(z, s) = Au(ax, a^2t)$$

is also a solution of the above heat equation for general parameters a, A . We choose $A = a^\alpha$, where we will determine α later. Now we have two different solutions $v(z, s)$ and $u(x, t)$ and we notice that

$$\frac{z^2}{s} = \frac{x^2}{t}, \quad \frac{v(z, s)}{s^{\alpha/2}} = \frac{u(x, t)}{t^{\alpha/2}}.$$

This suggests to us that there are solutions such that these quantities are **invariant** with respect to **dilation** and therefore we can define

$$\xi = \frac{x}{\sqrt{t}}, \quad f(\xi) = \frac{u(x, t)}{t^{\alpha/2}}.$$

Therefore we look for *similarity solution* in the form

$$u(x, t) = f\left(\frac{x}{\sqrt{t}}\right) t^{\alpha/2}. \quad (4.4)$$

Plugging it into the heat equation we obtain the following equation for f

$$\frac{\alpha}{2}f(\xi) - \frac{1}{2}\xi f'(\xi) - Df''(\xi) = 0. \quad (4.5)$$

Now we have a choice of α that will give use solution to the (4.5) and hence a solution to heat equation. We look for a solution that is integrable and decays at infinity together with its x -derivatives

$$\int_{\mathbb{R}} |u(x, t)| dx < \infty, \quad u_x(x, t) \rightarrow 0 \text{ for } x \rightarrow \pm\infty.$$

If we impose these assumptions then integrating heat equation over $\mathbb{R}$ we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} u(x, t) dx = D \int_{\mathbb{R}} u_{xx}(x, t) dx = D(u_x(\infty, t) - u_x(-\infty, t)) = 0.$$

Therefore

$$\int_{\mathbb{R}} u(x, t) dx = \text{const} \quad \text{for all } t > 0.$$

Without loss of generality (by rescaling u) we can choose the above constant to be 1. Plugging in our ansatz (4.4) for $u(x, t)$ we obtain

$$t^{\alpha/2} \int_{\mathbb{R}} f\left(\frac{x}{\sqrt{t}}\right) dx = 1,$$

or

$$t^{(\alpha+1)/2} \int_{\mathbb{R}} f(\xi) d\xi = 1 \quad \text{for all } t > 0.$$

Therefore we have to take $\alpha = -1$.

The equation (4.5) becomes

$$-\frac{1}{2}f(\xi) - \frac{1}{2}\xi f'(\xi) - Df''(\xi) = 0$$

and we can solve it as follows

$$\frac{1}{2}(\xi f(\xi))' + Df''(\xi) = \left(Df'(\xi) + \frac{1}{2}\xi f(\xi) \right)' = 0.$$

Using behavior at infinity we obtain

$$Df'(\xi) + \frac{1}{2}\xi f(\xi) = 0$$

and

$$f(\xi) = Ae^{-\frac{\xi^2}{4D}}.$$

Now recalling that $\int_{\mathbb{R}} f(\xi) d\xi = 1$ we have $A = \frac{1}{\sqrt{4\pi D}}$ and

$$u(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}. \quad (4.6)$$

The solution in (4.6) is called *the fundamental solution of heat equation*.

4.1.2 Properties of the fundamental solution

In what follows below we will use fundamental solution to construct solutions to heat equation. We will denote the fundamental solution as

$$\Phi(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

and note that it has the following properties

1. $\Phi_t = D\Phi_{xx}$ for all $x \in \mathbb{R}, t > 0$;
2. $\Phi(x, t) = \Phi(-x, t) > 0$, for all $x \in \mathbb{R}, t > 0$;
3. $\Phi(x, t)$ is a smooth (C^∞) function of (x, t) for $x \in \mathbb{R}, t > 0$;
4. $\int_{\mathbb{R}} \Phi(x, t) dx = 1$ for all $t > 0$.
5. As $t \rightarrow 0^+$, $\Phi(x, t) \rightarrow 0$ for all $x \neq 0$ and $\Phi(0, t) \rightarrow \infty$.

Looking at the properties 4 and 5 we note that as $t \rightarrow 0^+$, $\Phi(x, t) \rightarrow \delta(x)$, where $\delta(x)$ is a delta-function. Note that we don't specify the convergence here as δ -function is not a function in a usual sense. In fact the convergence will be in a sense of "measures" but we don't discuss it here.

4.1.3 The Delta Function

Let us consider the following sequence of functions

$$f_a(x) = \frac{e^{-x^2/a^2}}{a\sqrt{\pi}}$$

and see what are the properties of the limit as $a \rightarrow 0$. It is clear that

- if $x \neq 0$ then $f_a(x) \rightarrow 0$ as $a \rightarrow 0$;
- if $x = 0$ then $f_a(x) \rightarrow \infty$ as $a \rightarrow 0$;
- $\int_{\mathbb{R}} f_a(x) dx = 1$ for all a and hence $\lim_{a \rightarrow 0} \int_{\mathbb{R}} f_a(x) dx = 1$

Is it possible to define a limit of $f_a(x)$ as $a \rightarrow 0$? This is impossible for “ordinary functions” but can define such a limit as a “**generalised function**”. Then we say

$$\delta(x)'' = \lim_{a \rightarrow 0} f_a(x) = \lim_{a \rightarrow 0} \frac{e^{-x^2/a^2}}{a\sqrt{\pi}}.$$

Based on this example we define a “generalized function” that we call **δ -function** as follows

Defn: The delta function, $\delta(x)$, is defined by

- $\delta(x) = 0$ for $x \neq 0$;
- $\delta(0) = \infty$;
- $\int_a^b \delta(x) dx = 1$ for any $a < 0 < b$.

The limiting form of the Gaussian is **not the only definition** – there are many – of $\delta(x)$.

E.g. Take “top-hat” $f_{th}(x) = \begin{cases} 0, & |x| > a \\ 1/2a, & |x| < a \end{cases}$. Then $\int_{-\infty}^{\infty} f_{th}(x) dx = 1$ and clearly then the definition above is satisfied by writing

$$\delta(x) = \lim_{a \rightarrow 0} f_{th}(x)$$

Properties of the Delta Function

1. $\delta(x)$ is an even function.

Proof: $\delta(-x) = 0$ for $x \neq 0$ and

$$\int_a^b \delta(-x) dx = \int_{-b}^{-a} \delta(x) dx = 1, \quad a < 0 < b$$

2. Shifted δ -Function. (obvious) $\delta(x - c) = 0$ for $x \neq c$ and $\int_a^b \delta(x - c) dx = 1$ provided $a < c < b$. Otherwise integral is zero.
3. Sampling Property (V. Important). For any sufficiently smooth function $f(x)$ (i.e. f and all derivatives are continuous) then

$$\int_{-\infty}^{\infty} \delta(x - c) f(x) dx = f(c)$$

I.e. it “picks out” value of $f(x)$ at $x = c$.

Justification: Use $\delta(x) = \lim_{a \rightarrow 0} f_{th}(x)$.

Then, using mean value property we have

$$\int_{-\infty}^{\infty} \delta(x - c) f(x) dx = \lim_{a \rightarrow 0} \int_{c-a}^{c+a} \left(\frac{1}{2a} \right) f(x) dx = f(c)$$

NOTE: need $f(x)$ to be continuous

Relationship to discontinuous functions

Defn: The Heaviside function $H(x)$ is defined by

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

$H'(x) = 0$ for $x \neq 0$, but $H'(x)$ not defined for $x = 0$, whilst

$$\int_a^b H'(x) dx = H(b) - H(a) = 1, \quad \text{if } a < 0 < b$$

Suggests $H'(x) = \delta(x)$...

Justification: Let $H(x) = \lim_{a \rightarrow 0} h(x)$ where

$$h(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{x+a}{2a} & \text{for } |x| < a \\ 1 & \text{for } x > a \end{cases}$$

And so $h'(x) = f_{th}(x)$ (top-hat function). So limit as $a \rightarrow 0$ gives

$$H'(x) = \delta(x)$$

Notes:

$$1. \operatorname{sgn}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases} \quad (\text{the signum function}). \text{ Then}$$

$$H(x) = \frac{1}{2}(1 + \operatorname{sgn}(x)), \quad \text{and} \quad \frac{d}{dx} [\operatorname{sgn}(x)] = 2H'(x) = 2\delta(x)$$

2. Can we differentiate $\delta(x)$? Yes - but not needed here.

4.1.4 Homogeneous heat equation

In this section we solve the following problem

$$u_t = Du_{xx}, \quad x \in \mathbb{R}, t > 0, \quad (4.7)$$

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}. \quad (4.8)$$

For simplicity, we assume that $\phi(x)$ is uniformly continuous and bounded function. The result is true for any integrable function $\phi(x)$.

We note that if $\Phi(x, t)$ is the fundamental solution of heat equation then for any fixed $y \in \mathbb{R}$ the function $\Phi(x - y, t)$ also solves heat equation (4.7). Moreover we can take a convolution of Φ and ϕ and obtain that

$$v(x, t) = \int_{\mathbb{R}} \Phi(x - y, t) \phi(y) dy$$

satisfies (4.7). Recalling that as $t \rightarrow 0$, $\Phi(x - y, t) \rightarrow \delta(x - y)$ we expect that

$$v(x, t) \rightarrow \int_{\mathbb{R}} \delta(x - y) \phi(y) dy \quad \text{as } t \rightarrow 0.$$

Recalling properties of δ -function we have $\int_{\mathbb{R}} \delta(x - y) \phi(y) dy = \phi(x)$ and therefore $v(x, 0) = \phi(x)$. It seems that if we take

$$u(x, t) = \int_{\mathbb{R}} \Phi(x - y, t) \phi(y) dy,$$

then it will be a solution of the problem (4.7), (4.8). Let's verify it.

It is clear that $u(x, t) = \int_{\mathbb{R}} \Phi(x - y, t) \phi(y) dy$ satisfies (4.7) as

$$u_t - Du_{xx} = \int_{\mathbb{R}} [\Phi_t(x - y, t) - D\Phi_{xx}(x - y, t)] \phi(y) dy = 0.$$

We just need to check that

$$u(x, t) \rightarrow \phi(x) \quad \text{as } t \rightarrow 0^+.$$

Using property 4 of Φ it is clear that

$$u(x, t) - \phi(x) = \int_{\mathbb{R}} \Phi(x - y, t) (\phi(y) - \phi(x)) dy.$$

As $\phi(x)$ is a uniformly continuous function we know that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|\phi(x) - \phi(y)| < \epsilon$ for all $|x - y| < \delta$. We now split our integral as follows

$$\int_{\mathbb{R}} \Phi(x - y, t)(\phi(y) - \phi(x)) dy = \int_{|y-x|<\delta} \Phi(x - y, t)(\phi(y) - \phi(x)) dy + \int_{|y-x|\geq\delta} \Phi(x - y, t)(\phi(y) - \phi(x)) dy. \quad (4.9)$$

It is clear that

$$\begin{aligned} \left| \int_{|y-x|<\delta} \Phi(x - y, t)(\phi(y) - \phi(x)) dy \right| &\leq \int_{|y-x|<\delta} \Phi(x - y, t)|\phi(y) - \phi(x)| dy \leq \\ &\epsilon \int_{|y-x|<\delta} \Phi(x - y, t) dy \leq \epsilon \int_{\mathbb{R}} \Phi(x - y, t) dy = \epsilon \end{aligned} \quad (4.10)$$

For the second integral we notice that as $\phi(x)$ is bounded, i.e. $|\phi(x)| \leq C$ for all $x \in \mathbb{R}$ then

$$\left| \int_{|y-x|\geq\delta} \Phi(x - y, t)(\phi(y) - \phi(x)) dy \right| \leq 2C \int_{|y-x|\geq\delta} \Phi(x - y, t) dy = 4C \int_{\delta}^{\infty} \Phi(z, t) dz.$$

Using explicit expression for Φ we can easily compute

$$\int_{\delta}^{\infty} \Phi(z, t) dz = \frac{1}{\sqrt{4\pi Dt}} \int_{\delta}^{\infty} e^{-\frac{z^2}{4Dt}} dz = \frac{1}{\sqrt{\pi}} \int_{\frac{\delta}{\sqrt{4Dt}}}^{\infty} e^{-s^2} ds \leq \frac{1}{2\sqrt{\pi}} e^{-\frac{\delta}{\sqrt{4Dt}}}.$$

Combining everything we obtain that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_{\mathbb{R}} \Phi(x - y, t)(\phi(y) - \phi(x)) dy \right| \leq \epsilon + \frac{2C}{\sqrt{\pi}} e^{-\frac{\delta}{\sqrt{4Dt}}}.$$

Taking limit as $t \rightarrow 0$ we have

$$\lim_{t \rightarrow 0^+} \left| \int_{\mathbb{R}} \Phi(x - y, t)(\phi(y) - \phi(x)) dy \right| \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we take $\epsilon \rightarrow 0^+$ to obtain

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}} \Phi(x - y, t)\phi(y) dy = \phi(x).$$

We just showed that $u(x, t) = \int_{\mathbb{R}} \Phi(x - y, t)\phi(y) dy$ solves (4.7), (4.8).

4.1.5 Heat equation with $f(x, t) \neq 0$ and zero initial data

In this section we solve the following problem

$$u_t = Du_{xx} + f(x, t), \quad x \in \mathbb{R}, t > 0, \quad (4.11)$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}. \quad (4.12)$$

We will use *Duhamel's principle* to find the solution of the above problem. Let us start with the following auxiliary problem. Fix any $s > 0$ and consider

$$v_t = Dv_{xx}, \quad x \in \mathbb{R}, t > s, \quad (4.13)$$

$$v(x, s) = f(x, s), \quad x \in \mathbb{R}. \quad (4.14)$$

Due to translation invariance of the heat equation we can easily check that for each fixed s

$$v(x, t; s) = \int_{\mathbb{R}} \Phi(x - y, t - s) f(y, s) dy$$

solves the problem (4.13), (4.14) (note that $v(x, s; s) = f(x, s)$). Now we define the following function

$$w(x, t) = \int_0^t v(x, t; s) ds$$

and claim that it solves (4.11), (4.12). Let's verify it.

It is clear that $w(x, 0) = 0$ and therefore (4.12) is satisfied. Now

$$w_t(x, t) = v(x, t; t) + \int_0^t v_t(x, t; s) ds = f(x, t) + \int_0^t v_t(x, t; s) ds$$

and

$$w_{xx}(x, t) = \int_0^t v_{xx}(x, t; s) ds.$$

Therefore for all $t > 0$

$$w_t - Dw_{xx} = f(x, t) + \int_0^t (v_t(x, t; s) - Dv_{xx}(x, t; s)) ds = f(x, t).$$

We just showed that

$$u(x, t) = \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) f(y, s) dy ds$$

solves (4.11), (4.12).

4.1.6 General problem

Now we are ready to solve the following general problem

$$u_t = Du_{xx} + f(x, t), \quad \text{for } x \in \mathbb{R}, t > 0 \quad (4.15)$$

$$u(x, 0) = \phi(x), \quad \text{for } x \in \mathbb{R}. \quad (4.16)$$

It is straightforward to check that if $v(x, t)$ satisfies

$$v_t = Dv_{xx}, \quad \text{for } x \in \mathbb{R}, t > 0 \quad (4.17)$$

$$v(x, 0) = \phi(x), \quad (4.18)$$

and $w(x, t)$ satisfies

$$w_t = Dw_{xx} + f(x, t), \quad \text{for } x \in \mathbb{R}, t > 0 \quad (4.19)$$

$$w(x, 0) = 0, \quad (4.20)$$

then $u(x, t) = v(x, t) + w(x, t)$ solves (4.15), (4.16). Since we already know from the previous subsections how to solve the problems for $v(x, t)$ and $w(x, t)$ we obtain

$$u(x, t) = \int_{\mathbb{R}} \Phi(x - y) \phi(y) dy + \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) f(y, s) dy ds.$$

4.2 Diffusion Equation on Semi-Infinite Domain

Problem:

$$u_t = Du_{xx} \text{ and } u(x, 0) = \phi(x) \text{ both for } x > 0$$

Need BC at $x = 0$...

4.2.1 Homogeneous Dirichlet BC at $x = 0$

I.e. Let $u(0, t) = 0$ for $t > 0$.

$$\text{Let } \Phi(x) = \begin{cases} \phi(x), & x > 0 \\ -\phi(-x), & x < 0 \end{cases}$$

I.e. $\Phi(x)$ is the *odd* extension of $\phi(x)$ into $x < 0$.

Now consider the extension of the PDE to all x : $U_t = DU_{xx}$ with $U(x, 0) = \Phi(x)$ both for $-\infty < x < \infty$.

Solution is from before

$$\begin{aligned} U(x, t) &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \Phi(\xi) e^{-\frac{(x-\xi)^2}{4Dt}} d\xi \\ &= \frac{1}{\sqrt{4\pi Dt}} \left[\int_{-\infty}^0 + \int_0^{\infty} \right] \\ &= \frac{1}{\sqrt{4\pi Dt}} \int_0^{\infty} \phi(\xi) \left[e^{-\frac{(x-\xi)^2}{4Dt}} - e^{-\frac{(x+\xi)^2}{4Dt}} \right] d\xi \end{aligned}$$

where in the negative integral, a change of vars, $\xi \rightarrow -\xi$, has been applied, and oddness, $\Phi(-\xi) = -\phi(\xi)$ used.

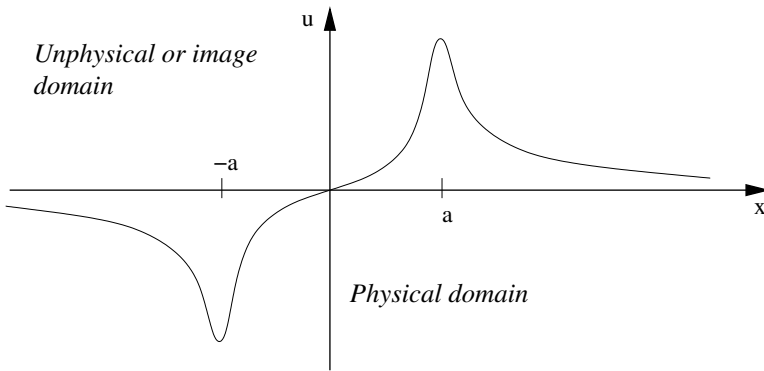
From eqn, clearly $U(0, t) = 0$. So U satisfies all of the conditions required by the problem for u (the PDE, the IC in $x > 0$ and the BC at $x = 0$) and so $u(x, t) = U(x, t)$ for $x > 0$.

Note: $U(x, t)$ extends the solution into the *unphysical domain* $x < 0$.

Example: $\phi(x) = \delta(x - a)$.

$$u(x, t) = \frac{e^{-\frac{(x-a)^2}{4Dt}} - e^{-\frac{(x+a)^2}{4Dt}}}{\sqrt{4\pi Dt}}.$$

This is fundamental solution for diffusion eqn. on half-line with zero B.C.



4.2.2 Homogeneous Neumann Condition at $x = 0$

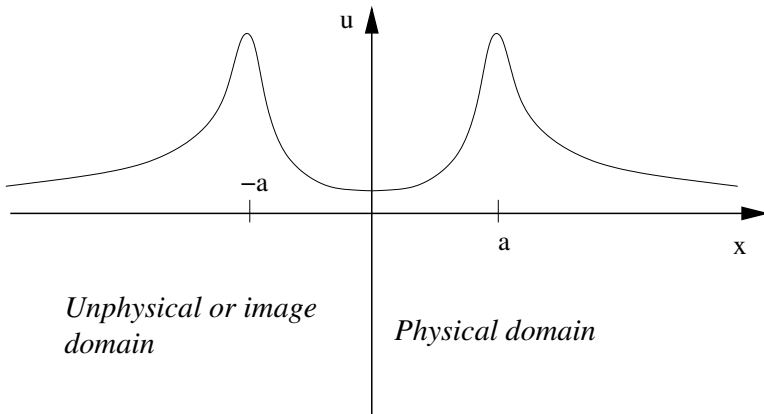
Similar argument using *even* extension of ϕ gives soln of diffusn eqn for $x > 0$ with $u_x(0, t) = 0$:

$$U(x, t) = \frac{1}{\sqrt{4\pi Dt}} \int_0^\infty \phi(\xi) \left[e^{-\frac{(x-\xi)^2}{4Dt}} + e^{-\frac{(x+\xi)^2}{4Dt}} \right] d\xi$$

Easy to check the B.C. holds. Hence $u(x, t) = U(x, t)$ in the physical domain, $x > 0$.

Example: $\phi(x) = \delta(x - a)$.

$$u(x, t) = \frac{e^{-\frac{(x-a)^2}{4Dt}} + e^{-\frac{(x+a)^2}{4Dt}}}{\sqrt{4\pi Dt}}.$$



4.3 Wave equation on $\mathbb{R}$. Method of fundamental solution.

Our goal is to solve the following problem

$$u_{tt} = c^2 u_{xx} + f(x, t), \quad x \in \mathbb{R}, t > 0 \quad (4.21)$$

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}. \quad (4.22)$$

We start by solving homogeneous problem with $f(x, t) \equiv 0$.

4.3.1 Homogeneous wave equation. D'Alembert's solution.

In this section we solve the following problem

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, t > 0, \quad (4.23)$$

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}. \quad (4.24)$$

Using the following change of variables $\xi = x - ct$, $\eta = x + ct$ we obtain

$$\begin{aligned} u_t &= -cu_\xi + cu_\eta, \quad u_x = u_\xi + u_\eta, \\ u_{tt} &= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}, \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}. \end{aligned}$$

Therefore

$$u_{tt} - c^2 u_{xx} = -2c^2 u_{\xi\eta} = 0.$$

It follows that $u = f(\xi) + g(\eta)$ and therefore returning to original variables

$$u(x, t) = f(x - ct) + g(x + ct).$$

Using initial data we obtain

$$f(x) + g(x) = \phi(x), \quad -cf'(x) + cg'(x) = \psi(x), \quad \text{for all } x \in \mathbb{R}.$$

After straightforward calculation it follows that for all $x \in \mathbb{R}$ and some fixed $a \in \mathbb{R}$

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2c} \int_a^x \psi(s) ds, \quad g(x) = \frac{1}{2}\phi(x) + \frac{1}{2c} \int_a^x \psi(s) ds.$$

Now we see that

$$u(x, t) = f(x - ct) + g(x + ct) = \frac{1}{2}\phi(x - ct) - \frac{1}{2c} \int_a^{x-ct} \psi(s) ds + \frac{1}{2}\phi(x + ct) + \frac{1}{2c} \int_a^{x+ct} \psi(s) ds.$$

Finally we obtain D'Alembert's solution

$$u(x, t) = \frac{1}{2}\phi(x - ct) + \frac{1}{2}\phi(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds. \quad (4.25)$$

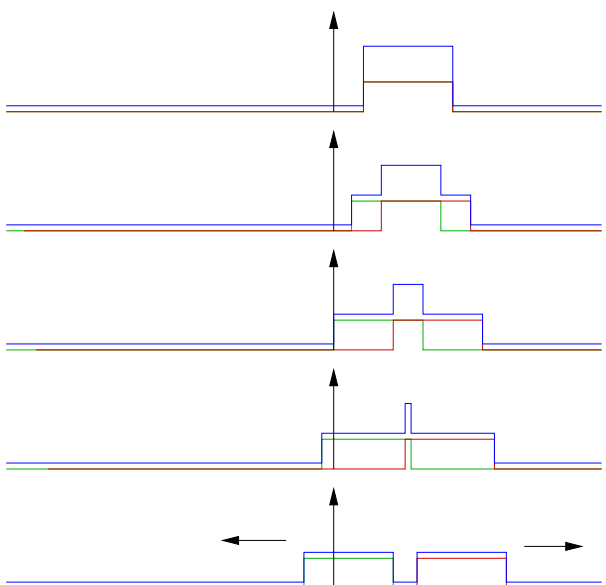
Example 1. $u(x, 0) = \phi(x) \equiv \begin{cases} 1 & \text{for } \alpha < x < \beta \\ 0 & \text{for otherwise} \end{cases}$,

and $u_t(x, 0) = 0$.

Since $\psi \equiv 0$, the integral vanishes and so

$$u(x, t) = \frac{1}{2}\phi(x - ct) + \frac{1}{2}\phi(x + ct)$$

This equation tells you that the initial fn. ϕ splits into two halves, each of height $\frac{1}{2}$ which move apart in opposite directions with speed c .



Rule: Blue = Green + Red

Example 2. $u(x, 0) \equiv 0, u_t(x, 0) = \psi(x)$.

In this case $\phi \equiv 0$, and so

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\xi) d\xi$$

Now consider the function: $\psi(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$.

This corresponds to an initial impulse or 'hammer blow' to the string across the range $|x| < a$.

The best way to approach this is as follows. Let

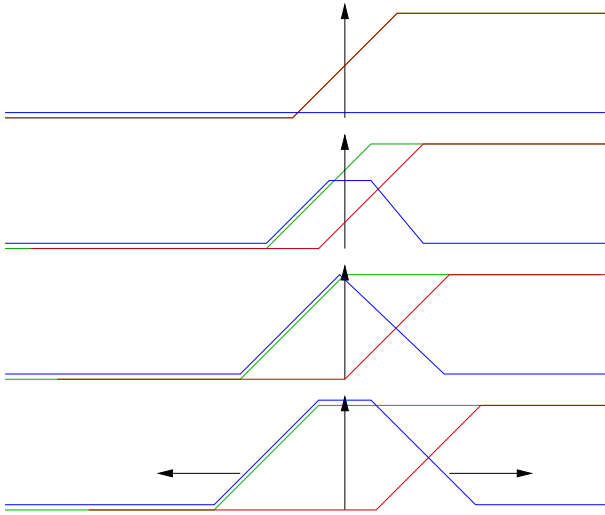
$$\Psi(x) = \frac{1}{2c} \int_{-\infty}^x \psi(\xi) d\xi = \begin{cases} 0 & \text{for } x < -a \\ (a+x)/2c & \text{for } -a < x < a \\ (2a)/2c & \text{for } x > a \end{cases}$$

which we can establish by careful consideration of the integral.

Then it is clear that

$$u(x, t) = \Psi(x + ct) - \Psi(x - ct)$$

Need pictures to see how this wave evolves. Two linear ramps, the green one moves to the left with increasing time at a speed c and represents the first term above and the red one goes right with speed c . The solution (blue) is the value of the red line subtracted from the value of the green line.



Rule: Blue = Green–Red

4.3.2 Wave equation with $f(x, t) \neq 0$ and zero initial data

In this section we solve the following problem

$$u_{tt} = c^2 u_{xx} + f(x, t), \quad x \in \mathbb{R}, t > 0, \quad (4.26)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0 \quad x \in \mathbb{R}. \quad (4.27)$$

We again use *Duhamel's principle* to find the solution of the above problem. Let us start with the following auxiliary problem. Fix any $s > 0$ and consider

$$v_{tt} = c^2 v_{xx}, \quad x \in \mathbb{R}, t > s, \quad (4.28)$$

$$v(x, s) = 0, \quad v_t(x, s) = f(x, s), \quad x \in \mathbb{R}. \quad (4.29)$$

Due to translation invariance of the wave equation we can easily check that for each fixed s

$$v(x, t; s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(r, s) dr.$$

solves the problem (4.28), (4.29) (note that $v(x, s; s) = 0$, $v_t(x, s; s) = f(x, s)$). Now we define the following function

$$w(x, t) = \int_0^t v(x, t; s) ds$$

and claim that it solves (4.26), (4.27). Let's verify it.

It is clear that $w(x, 0) = 0$ and therefore (4.27) is satisfied. Now

$$w_t(x, t) = v(x, t; t) + \int_0^t v_t(x, t; s) ds = \int_0^t v_t(x, t; s) ds,$$

$$w_{tt}(x, t) = v_{tt}(x, t; t) + \int_0^t v_{tt}(x, t; s) ds = f(x, t) + \int_0^t v_{tt}(x, t; s) ds.$$

and

$$w_{xx}(x, t) = \int_0^t v_{xx}(x, t; s) ds.$$

Therefore for all $t > 0$

$$w_{tt} - c^2 w_{xx} = f(x, t) + \int_0^t (v_{tt}(x, t; s) - c^2 v_{xx}(x, t; s)) ds = f(x, t).$$

We just showed that

$$u(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(r, s) dr ds$$

solves (4.26), (4.27).

4.3.3 General problem

Now we are ready to solve the following general problem

$$u_{tt} = c^2 u_{xx} + f(x, t), \quad \text{for } x \in \mathbb{R}, t > 0 \quad (4.30)$$

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \quad \text{for } x \in \mathbb{R}. \quad (4.31)$$

It is straightforward to check that if $v(x, t)$ satisfies

$$v_{tt} = c^2 v_{xx}, \quad \text{for } x \in \mathbb{R}, t > 0 \quad (4.32)$$

$$v(x, 0) = \phi(x), \quad v_t(x, 0) = \psi(x) \quad \text{for } x \in \mathbb{R}. \quad (4.33)$$

and $w(x, t)$ satisfies

$$w_{tt} = c^2 w_{xx} + f(x, t), \quad \text{for } x \in \mathbb{R}, t > 0 \quad (4.34)$$

$$w(x, 0) = 0, \quad w_t(x, 0) = 0 \quad \text{for } x \in \mathbb{R} \quad (4.35)$$

then $u(x, t) = v(x, t) + w(x, t)$ solves (4.30), (4.31). Since we already know from the previous subsections how to solve the problems for $v(x, t)$ and $w(x, t)$ we obtain

$$u(x, t) = \frac{1}{2} \phi(x - ct) + \frac{1}{2} \phi(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(r) dr + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(r, s) dr ds.$$

4.4 Reflection of Waves

Consider $u_{tt} = c^2 u_{xx}$ for $x > 0$ with the B.C. at $x = 0$ given by $u(0, t) = 0$ for $t > 0$.

I.C's are $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$, for $x > 0$.

This is a semi-infinite problem as in §4.6.1. Use the same solution trick as then:

I.e. let $\Phi(x) = \begin{cases} \phi(x), & x > 0, \\ -\phi(-x), & x < 0 \end{cases}$ and $\Psi(x) = \begin{cases} \psi(x), & x > 0, \\ -\psi(-x), & x < 0 \end{cases}$ be the odd extensions of the I.Cs into $x < 0$.

So let $U(x, t)$ satisfy $U_{tt} = c^2 U_{xx}$ in $-\infty < x < \infty$ with $U(x, 0) = \Phi(x)$ and $U_t(x, 0) = \Psi(x)$. Then from previous section, solution is

$$U(x, t) = \frac{1}{2}[\Phi(x - ct) + \Phi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \Psi(\xi) d\xi$$

Now

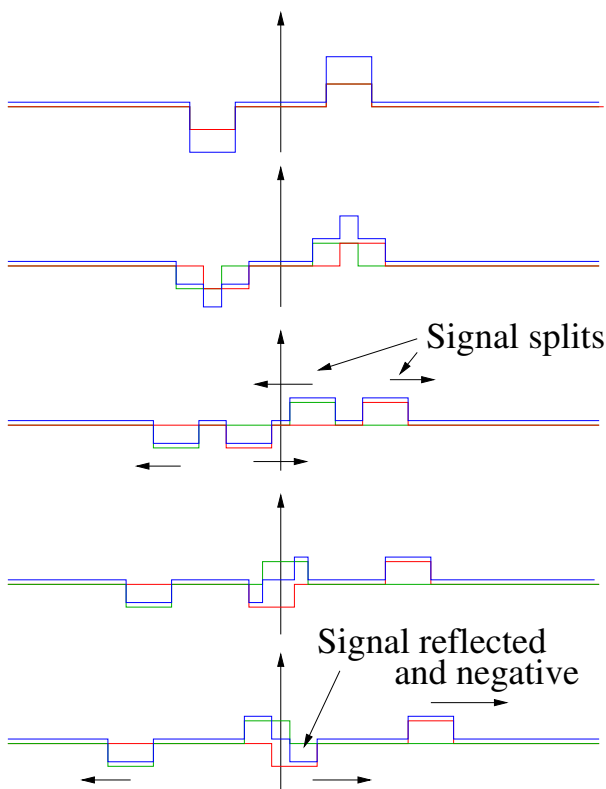
$$U(0, t) = \frac{1}{2}[\Phi(-ct) + \Phi(ct)] + \frac{1}{2c} \int_{-ct}^{ct} \Psi(\xi) d\xi = 0$$

(after using the oddness of the functions).

Since $U(x, t)$ satisfies the same PDE, ICs as $u(x, t)$ in $x > 0$ and the BC at $x = 0$ it follows that $u(x, t) = U(x, t)$ in $x > 0$.

Example: $\phi(x) = \begin{cases} 1 & \text{for } 1 < x < 2 \\ 0 & \text{for } 0 < x < 1 \text{ or } x > 2 \end{cases}$, $\psi(x) \equiv 0$. Take the odd extension of ϕ , the first graph on the left below. Then $U(x, t) = \frac{1}{2}[\Phi(x - ct) + \Phi(x + ct)]$

The series of pictures are increasing in time. The pictures show the two separate parts of u (red and green) and the blue line is the sum of those two parts. It can be seen that the reflection at the wall flips the wave over and reflects it back towards the right.



Rule: Blue = Green + Red

Chapter 5

PDEs on Infinite Domains: Fourier Transforms

5.1 The Fourier Transform

5.1.1 Definitions

Defn: The **Fourier transform** (F.T.) of $f(x)$ is

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Notes:

1. Sometimes, there is a factor of $1/\sqrt{2\pi}$ in the definition; sometimes e^{-ikx} is e^{ikx} . This is OK, as long as *consistent* with the definition of the **inverse**. Also, sometimes $\hat{f}$ written as $\bar{f}(k)$ or $F(k)$.
2. Integral only exists if $f(x) \rightarrow 0$ “fast enough” as $x \rightarrow \pm\infty$.
3. Can think of F.T. as an operation: $\hat{f} = \mathcal{F}\{f\}$; $\mathcal{F}$ is a **linear operator**.

Example 1. Take $f(x) = e^{-a|x|}$, with $a > 0$. Then

$$f(x) = \begin{cases} e^{ax} & \text{for } x < 0 \\ e^{-ax} & \text{for } x > 0 \end{cases} \quad \text{Taking the F.T.:}$$

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^0 e^{(a-ik)x} dx + \int_0^{\infty} e^{(-a-ik)x} dx \\ &= \left[\frac{1}{a-ik} + \frac{1}{a+ik} \right] \\ &= \left(\frac{2a}{a^2 + k^2} \right). \end{aligned}$$

Example 2. Take $f(x) = e^{-ax^2}$, with $a > 0$. Then taking the F.T.:

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx = e^{-\frac{k^2}{4a}} \int_{-\infty}^{\infty} e^{-a(x + \frac{ik}{2a})^2} dx = \sqrt{\frac{\pi}{a}} e^{-\frac{k^2}{4a}}$$

5.1.2 Simple Properties

1. $\mathcal{F}$ is a linear operator: For fns f, g and consts a, b we have
 $\mathcal{F}\{af + bg\} = a\mathcal{F}\{f\} + b\mathcal{F}\{g\}$
2. If $f(x)$ is even in x , then $\hat{f}(k)$ is even in k .

Proof:

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_{-\infty}^{\infty} f(-x) e^{ikx} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-i(-k)x} dx \\ &= \hat{f}(-k) \end{aligned}$$

where $f(x) = f(-x)$ and a change of variables $x \rightarrow -x$ have been used.

Similarly, if $f(x)$ is odd then $\hat{f}(k)$ is odd also.

3. Furthermore, if f is real & even, then $\hat{f}$ is real.

Proof:

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^0 f(-x) e^{-ikx} dx + \int_0^{\infty} f(x) e^{-ikx} dx \\ &= 2 \int_0^{\infty} f(x) \cos(kx) dx \end{aligned}$$

Similarly, if $f(x)$ is real & odd, then $\hat{f}(k)$ is imaginary.

4. (V. Important) $\widehat{f'}(k) \equiv \mathcal{F}\left\{\frac{df}{dx}\right\} = ik\hat{f}(k)$. This is key result for application of F.T.'s to PDEs.

Proof:

$$\begin{aligned} \widehat{f'}(k) &= \int_{-\infty}^{\infty} \frac{df}{dx} e^{-ikx} dx \\ &= \left[f(x) e^{-ikx} \right]_{-\infty}^{\infty} + ik \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ &= ik\hat{f}(k) \end{aligned}$$

by integration by parts and using $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$.

5. (Shift Property) If $g(x) = f(x + a)$, then $\hat{g}(k) = e^{ika} \hat{f}(k)$.

Proof:

$$\hat{g}(k) = \int_{-\infty}^{\infty} f(x + a) e^{-ikx} dx = \int_{-\infty}^{\infty} f(x') e^{ika} e^{-ikx'} dx' = e^{ika} \hat{f}(k)$$

After using $x' = x + a$.

5.1.3 Products and Convolutions

$\mathcal{F}\{fg\} \neq \mathcal{F}\{f\}\mathcal{F}\{g\}$, it's more complicated.

Defn: The **convolution** of fns f, g is the fn $f * g$ defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(\xi)g(x - \xi) d\xi$$

Then $f * g = g * f$.

Proof: Easy.

$$(g * f)(x) = \int_{-\infty}^{\infty} g(\xi)f(x - \xi) d\xi = \int_{-\infty}^{\infty} g(x - \xi')f(\xi') d\xi'$$

after a change of variables $\xi' = x - \xi$.

Transforms of Products and Convolutions

$$\mathcal{F}\{f * g\} \equiv \hat{f} * \hat{g} = \hat{f}\hat{g}$$

Proof: Start with the L.H.S. of above:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-ikx} \left[\int_{-\infty}^{\infty} f(\xi)g(x - \xi) d\xi \right] dx &= \int_{-\infty}^{\infty} f(\xi)e^{-ik\xi} \left[\int_{-\infty}^{\infty} g(x - \xi)e^{-ik(x - \xi)} dx \right] d\xi \\ &= \int_{-\infty}^{\infty} f(\xi)e^{-ik\xi} \left[\int_{-\infty}^{\infty} g(x')e^{-ikx'} dx' \right] d\xi \\ &= \hat{f}(k)\hat{g}(k) \end{aligned}$$

after switching order of integration and using a change of variable $x' = x - \xi$.

5.1.4 Inverse Fourier Transforms

Defn: The **inverse F.T.:**

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx} dk$$

Can think of this as $f(x) = \mathcal{F}^{-1}\{\hat{f}\}$ where $\mathcal{F}^{-1}$ is the inverse operator to $\mathcal{F}$.

Obviously it has to be shown that $f(x) = \mathcal{F}^{-1}\{\hat{f}\}$ is true.

Claim: If $f(x)$ is continous function decaying fast enough at $\pm\infty$ and $\hat{f}(k) = \int_{\mathbb{R}} f(x)e^{-ikx} dx$ then $f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(k)e^{ikx} dk$.

Proof: Let us define

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(k)e^{ikx} dk = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(z)e^{-ikz} dz \right) e^{ikx} dk = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(z)e^{-ik(z-x)} dz dk$$

We have to show that $g(x) = f(x)$ but it is really not clear what to do with this $g(x)$. Therefore we try to approximate this $g(x)$ by the following functions

$$g_\epsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) e^{-ik(z-x)} e^{-\epsilon k^2} dk dz = \frac{1}{2\pi} \int_{\mathbb{R}} f(z) \left(\int_{\mathbb{R}} e^{-ik(z-x) - \epsilon k^2} dk \right) dz.$$

It is clear that $g_\epsilon(x) \rightarrow g(x)$ as $\epsilon \rightarrow 0$. Using Example 2 we know that

$$\int_{\mathbb{R}} e^{-ik(z-x) - \epsilon k^2} dk = \sqrt{\frac{\pi}{\epsilon}} e^{-\frac{(x-z)^2}{4\epsilon}}.$$

Therefore it is clear

$$g_\epsilon(x) = \frac{1}{\sqrt{4\pi\epsilon}} \int_{\mathbb{R}} f(z) e^{-\frac{(x-z)^2}{4\epsilon}} dz.$$

But we already know (recall fundamental solutions) that

$$g_\epsilon(x) = \frac{1}{\sqrt{4\pi\epsilon}} \int_{\mathbb{R}} f(z) e^{-\frac{(x-z)^2}{4\epsilon}} dz \rightarrow f(x) \text{ as } \epsilon \rightarrow 0.$$

Therefore we have $g_\epsilon(x) \rightarrow g(x)$ and $g_\epsilon(x) \rightarrow f(x)$ which implies $g(x) = f(x)$. QED

Fourier Integral Theorem. If f is piecewise continuous and absolutely integrable $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ then integral for $\hat{f}(k)$ converges. At a point of discontinuity, $x = c$ say, $f(x)$ defined by its inverse F.T. converges to $\frac{1}{2}(f(c-) + f(c+))$ as in F.S.

Proof: too difficult for this course.

Example 3: Let's try to informally find the Fourier transform of $\delta(x)$

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = 1$$

So F.T. of $\delta(x)$ is unity !

Informal Fourier inversion formula gives

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk = \frac{1}{\pi} \int_0^{\infty} \cos kx dk \quad !!$$

since $\sin kx$ is odd in k and integrates to zero. It obviously does not make much sense as this integral does not converge. In order to make sense out of this we have to deal with approximation of $\delta(x)$ rather than $\delta(x)$, for instance:

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{4\pi\epsilon}} e^{-\frac{(x-z)^2}{4\epsilon}} = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx - \epsilon k^2} dk.$$

Therefore we "can" formally deal with Fourier transform of $\delta(x)$.

Transform of Shifted Delta Function

$$\mathcal{F}\{\delta(x-a)\} = \int_{-\infty}^{\infty} \delta(x-a) e^{-ikx} dx = e^{-ika}$$

(Consistent with the general property of FTs that if $g(x) = f(x + a)$ then $\hat{g} = e^{ika} \hat{f}$.)

Inverse F.T.

$$\delta(x - a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-a)} dk$$

Transform of Complex Exponential

$$\mathcal{F}\{e^{iax}\} = \int_{-\infty}^{\infty} e^{-i(k-a)x} dx = 2\pi\delta(k - a)$$

Example 4. Inverse F.T. says that (see Example 1)

$$f(x) = e^{-a|x|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + k^2} e^{ikx} dk$$

If we let $x \rightarrow -k$ and $k \rightarrow x$ simultaneously, we get

$$\frac{\pi}{a} e^{-a|k|} = \int_{-\infty}^{\infty} \frac{1}{a^2 + x^2} e^{-ikx} dx$$

I.e. if $f(x) = 1/(a^2 + x^2)$ then $\hat{f}(k) = \pi e^{-a|k|}/a$.

We get this for free – it comes from the inverse FT – and we don't need to confirm it. But can we? Yes – using complex function theory and integration in the complex plane.

We want to calculate

$$\int_{-\infty}^{\infty} g(x) dx$$

where integrand is

$$g(x) = \frac{e^{-ikx}}{a^2 + x^2} = \frac{e^{-ikx}}{(x - ia)(x + ia)}$$

We extend x to complex values, so there are **simple poles** (the denominator vanishes with multiplicity one) at $x = \pm ia$.

To compute $\int_{-\infty}^{\infty} g(x) dx$, form a **closed contour** in the complex x -plane which includes the real x -axis. Why? Because **Cauchy's residue theorem** (CRT) states that “the value of the integral round a closed contour is equal to $2\pi i$ times the sum of the residues at the poles inside the contour.”

Residue at $x = ia$ is $\lim_{x \rightarrow ia} (x - ia)g(x) = \frac{e^{-ik(ia)}}{2ia}$.

Residue at $x = -ia$ is $\lim_{x \rightarrow -ia} (x + ia)g(x) = \frac{e^{-ik(-ia)}}{-2ia}$.

How to close the contour?

Two (sensible) possibilities: Either close contour with a large semi-circle C_+ in the upper-half complex x -plane or C_- in the lower-half plane. Which way to go?

If we choose C_+ then CRT says

$$\int_{-\infty}^{\infty} g(x) dx + \int_{C_+} g(x) dx = 2\pi i \text{Res}(+ia)$$

and if C_- we choose

$$\int_{-\infty}^{\infty} g(x) dx + \int_{C_-} g(x) dx = -2\pi i \text{Res}(-ia)$$

(the minus sign accounts for the fact that the loop is in a **clockwise** direction – CRT assumes anticlockwise)... provided the C_+ , C_- make sense.

Which one ? Parametrise large semi-circle $C_{\pm}$ by $x = Re^{i\theta}$ as $R \rightarrow \infty$. Then

$$\exp\{-ikx\} = \exp\{-ikRe^{i\theta}\} = \exp\{-iRk \cos \theta\} \exp\{Rk \sin \theta\}$$

- If $k > 0$ then choice $-\pi < \theta < 0$ means above tends to zero as $R \rightarrow \infty$. I.e. go into **lower** half plane,

$$\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a^2 + x^2} dx + 0 = -2\pi i \frac{e^{-ak}}{-2ia} = \frac{\pi}{a} e^{-ak}$$

- If $k < 0$ then if $0 < \theta < \pi$ means above tends to zero as $R \rightarrow \infty$. I.e. go into **upper** half plane.

$$\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a^2 + x^2} dx + 0 = 2\pi i \frac{e^{ak}}{-2ia} = \frac{\pi}{a} e^{ak}$$

This confirms the result !

Example 5: Take $f(x) = e^x$. F.T. does not exist because the integral not convergent. Same for e^{-x} .

Example 6: Take $f_{th}(x) = \begin{cases} 1/2a & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$.

This is called a “**top-hat**” function. Then

$$\hat{f}_{th}(k) = \int_{-a}^a \frac{1}{2a} e^{-ikx} dx = \left[\frac{e^{ika} - e^{-ika}}{2ika} \right] = \frac{\sin ka}{ka}$$

5.2 Heat Equation on an ∞ -Domain

Consider

$$u_t = Du_{xx} \quad \text{for} \quad -\infty < x < \infty$$

with an I.C. of

$$u(x, 0) = \phi(x), \quad \phi(x) \text{ a given function}$$

Let $\hat{u}(k, t) = \int_{-\infty}^{\infty} u(x, t) e^{-ikx} dx$ (i.e. $\hat{u}(k, t)$ is the F.T. of $u(x, t)$ w.r.t. x).

Take the F.T. of P.D.E. (i.e. multiply $u_t = Du_{xx}$ by e^{-ikx} and integrate over $-\infty < x < \infty$). Then

$$\mathcal{F}\{u_t\} = D\mathcal{F}\{u_{xx}\}$$

or

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-ikx} dx = D \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-ikx} dx$$

and so

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} u e^{-ikx} dx = D(ik)\mathcal{F}\{u_x\} = D(ik)^2 \mathcal{F}\{u\}$$

which gives

$$\hat{u}_t = -Dk^2 \hat{u}$$

because $\partial/\partial t$ does not interfere with the F.T. in x and using the property of F.T.'s of derivatives.

This is just a 1st order O.D.E. in t .

Solution: $\hat{u}(k, t) = Ce^{-k^2 Dt}$

for the "integration constant", $C \equiv C(k)$ but independent of t .

To determine $C(k)$ need extra info... the I.C. is what we need (there are no 'boundaries' in this problem, although $\pm\infty$ may be regarded as boundaries).

Taking the F.T. of the I.C. gives

$$\hat{u}(k, 0) = \hat{\phi}(k)$$

as before, trivial O.D.E. in t with I.C. fixing the integration "constant".

I.e. Solution in **Transform space** is:

$$\hat{u}(k, t) = \hat{\phi}(k)e^{-Dk^2 t}.$$

Inversion ? For specific $\phi(x)$, will know $\hat{\phi}(k)$ and may be able to invert directly.

Here we don't have a specific $\hat{\phi}(k)$... Can use **convolution** because $\hat{u}$ is in the form of a **product** of two transform functions of k . All that's needed for convolution to work is that we know the functions that give the two F.T.'s in the product.

Well $\mathcal{F}\{\phi(x)\} = \hat{\phi}(k)$, so that's easy.

Using Example 2 we have

$$\mathcal{F}\left\{\frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}\right\} = e^{-k^2 Dt}$$

So **convolution theorem** gives

$$u(x, t) = \mathcal{F}^{-1}\{\hat{\phi}(k)e^{-k^2 Dt}\} = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \phi(\xi) e^{-\frac{(x-\xi)^2}{4Dt}} d\xi$$

5.2.1 Heat equation with the right hand side

Consider

$$u_t = Du_{xx} + f(x, t) \quad \text{for} \quad -\infty < x < \infty$$

with an I.C. of

$$u(x, 0) = \phi(x),$$

$$\text{Let } \hat{u}(k, t) = \int_{-\infty}^{\infty} u(x, t) e^{-ikx} dx$$

Take the F.T. of P.D.E. (i.e. multiply $u_t = Du_{xx} + f(x, t)$ by e^{-ikx} and integrate over $-\infty < x < \infty$). Then

$$\mathcal{F}\{u_t\} = D\mathcal{F}\{u_{xx}\} + \mathcal{F}\{f\}$$

or

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-ikx} dx = D \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-ikx} dx + \int_{-\infty}^{\infty} f(x, t) e^{-ikx} dx$$

and so

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} u e^{-ikx} dx = D(ik)^2 \mathcal{F}\{u\} + \mathcal{F}\{f\} = D(ik)^2 \mathcal{F}\{u\} + \mathcal{F}\{f\}$$

which gives

$$\hat{u}_t = -Dk^2 \hat{u} + \hat{f}$$

Initial condition gives $\hat{u}(k, 0) = \hat{\phi}(k)$ and therefore we are now solving the first order ODE with the right hand side. It is clear that

$$\hat{u}(k, t) = \hat{\phi}(k) e^{-Dk^2 t} + e^{-Dk^2 t} \int_0^t e^{Dk^2 s} \hat{f}(k, s) ds$$

Taking inverse FT we obtain

$$u(x, t) = \int_{\mathbb{R}} \Phi(x - y, t) \phi(y) dy + \int_0^t \int_{\mathbb{R}} \Phi(x - y, t - s) f(y, s) dy ds.$$

5.3 Particular Initial Conditions

5.3.1 Delta Function

If $u(x, 0) = \delta(x)$ (e.g. all the heat/chemical initially dumped at the origin – a decent mathematical model), then $\phi(x) = \delta(x)$

and

$$u(x, t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \delta(\xi) e^{-\frac{(x-\xi)^2}{4Dt}} d\xi = \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

Defn: This is the **fundamental solution** of the diffusion equation.

Properties of this solution

- Initially $u(x, 0) = \delta(x)$ and so $\int_{-\infty}^{\infty} u dx = 1$. This is the total amount of “stuff” in the system at $t = 0$.
- For each fixed t , $u(x, t)$ is a Gaussian in x , but spreads out with increasing t .
- For $t > 0$ we have $\int_{-\infty}^{\infty} u dx = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-x^2/4Dt} dx = \frac{1}{\sqrt{4\pi Dt}} \sqrt{4\pi Dt} = 1$. I.e. the amount of stuff in the system remains constant. Expected.
- For the I.C. to $u(x, 0) = \delta(x - a)$, easy to see solution is $u(x, t) = \frac{e^{-\frac{(x-a)^2}{4Dt}}}{\sqrt{4\pi Dt}}$

5.3.2 Heaviside Function

Assume $u(x, 0) = H(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$.

Note: Appears not to be valid, as one can't take the F.T. of the I.C. (it doesn't tend to zero at infinity), but can be made rigorous by taking limits.

From the General Solution,

$$u(x, t) = \frac{1}{\sqrt{4\pi Dt}} \int_0^{\infty} e^{-(x-\xi)^2/4Dt} d\xi$$

Change of variable: $s = (\xi - x)/\sqrt{4Dt}$. Then

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4Dt}}^{\infty} e^{-s^2} ds = \frac{1}{2} \text{erfc}(-x/\sqrt{4Dt})$$

where

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-s^2} ds$$

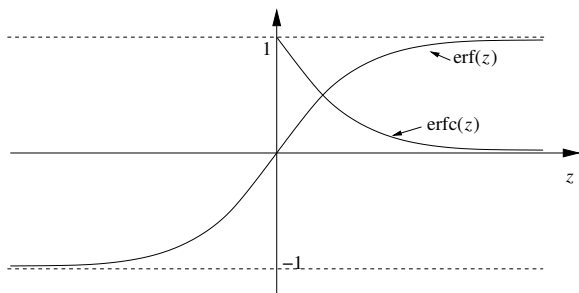
is a special function call the **Complementary Error Function**. Also, define

$$\text{erf}(z) = 1 - \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

as the **Error Function**. This relationship follows since

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = 1 \equiv \text{erf}(\infty)$$

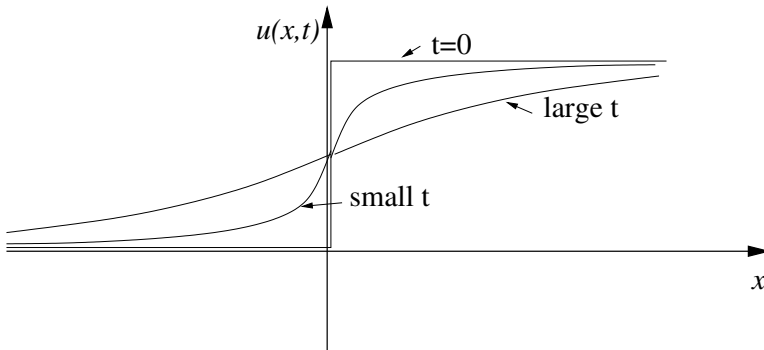
Note also that $\text{erf}(z) = -\text{erf}(-z)$ is an odd function whilst $\text{erf}(0) = 0$.



Using this information, gives

$$u(x, t) = \frac{1}{2}(1 + \operatorname{erf}(x/\sqrt{4Dt}))$$

as the solution with the Heaviside function as the I.C.



Final note: $\frac{d}{dx}(\operatorname{erf}(x/\sqrt{4Dt})) = \frac{e^{-x^2/4Dt}}{\sqrt{4Dt}}.$

5.3.3 Use of Superposition

The P.D.E. is *linear*, so can apply the principle of superposition. E.g. infinite domain with mass Q_a at $x = a$ and mass Q_b at $x = b$. Then solution is the sum of the solutions of the two parts separately:

$$u(x, t) = \frac{Q_a e^{-(x-a)^2/4Dt} + Q_b e^{-(x-b)^2/4Dt}}{\sqrt{4\pi Dt}}$$

using formula above (and interchanging the variables, $k \rightarrow -x$ and $x \rightarrow k$).

Chapter 6

First Order PDEs

6.1 The method of characteristics

6.1.1 The Simplest Case

Suppose $u(x, t)$ satisfies the PDE

$$au_t + bu_x = 0$$

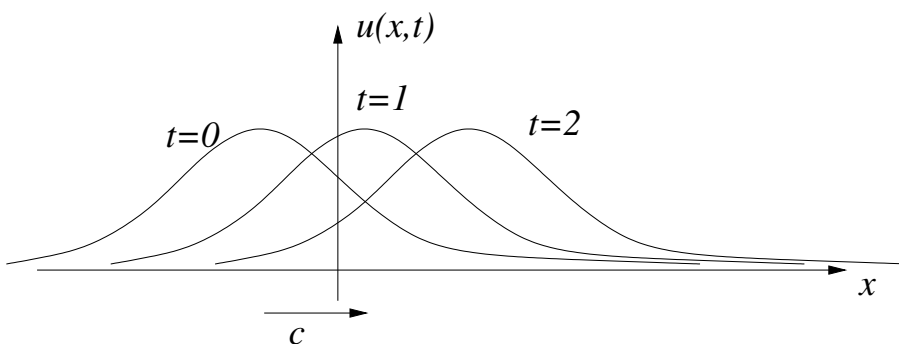
where a, b are constant.

If $a = 0$, the PDE is trivial (it says that $u_x = 0$ and so $u = f(t)$).

If $a \neq 0$, it reduces to

$$u_t + cu_x = 0 \text{ where } c = b/a. \quad (6.1)$$

We know from that the solution is $f(x - ct)$. This represents a wave travelling in the x direction with speed c , and with constant shape.



A new approach:

Assume $x = x(s)$ and $t = t(s)$ describes curves in the (x, t) -plane where s parametrises the curve.

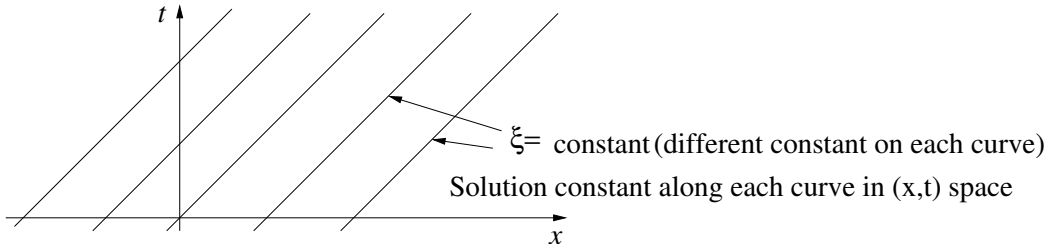
By the chain rule,

$$\frac{du}{ds} = \frac{dt}{ds}u_t + \frac{dx}{ds}u_x$$

So if $\frac{dt}{ds} = 1$ and $\frac{dx}{ds} = c$ then $\frac{du}{ds} = u_t + cu_x = 0$.

Then, integrating $t = s + \text{const}$, $x = cs + \text{const}$ and $u = \text{const}$ along one of these curves.

Elimating s gives $x = ct + \text{const}$ or $x = ct + \xi$, say. So along curves $x - ct = \xi$, a constant, $u = \text{const}$. In the (x, t) -plane these are straight lines of gradient $1/c$ passing through $x = \xi$.



Consider the IC $u(x, 0) = f(x)$, a given function. Then since $x = \xi$ at $t = 0$, $u = f(\xi)$ on $t = 0$ and so u must always equal $f(\xi)$ along the **characteristic curves** $\xi = x - ct$. Therefore $u(x, t) = f(\xi) = f(x - ct)$.

6.1.2 A more complicated example

Consider

$$u_t + xu_x = \sin t \quad \text{with I.C. } u(x, 0) = f(x).$$

- a 1st order **inhomogeneous linear** PDE with **non-constant** coefficients.

Use **method of characteristics** as before, in which (x, t) are mapped into variables (ξ, s) along which the PDE can be transformed into an ODE.

Specifically, we choose to set $x = \xi$ and $s = 0$ on $t = 0$.

By chain rule:

$$\frac{du}{ds} = \frac{dt}{ds}u_t + \frac{dx}{ds}u_x = \sin t$$

So the **characteristic equations** are:

$$\text{if (i): } \frac{dt}{ds} = 1, \quad \text{and (ii): } \frac{dx}{ds} = x, \quad \text{then (iii): } \frac{du}{ds} = \sin t,$$

Integrating up and imposing $x = \xi$, $s = 0$ on $t = 0$ gives

$$\text{(i): } t = s, \quad \text{(ii): } x = \xi e^s, \quad \text{and (iii): } \frac{du}{ds} = \sin s \Rightarrow \frac{du}{ds} = \sin s \Rightarrow u = -\cos s + C(\xi)$$

where $C(\xi)$ is a constant along each curve, but varies from curve to curve hence the dependence on ξ .

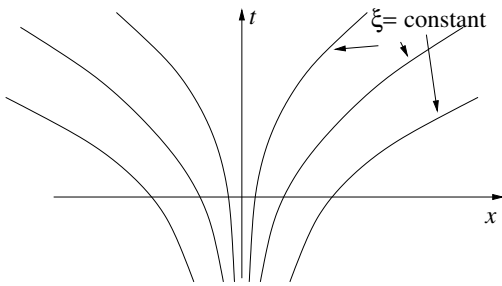
Since $x = \xi$ on $t = 0$, $u(\xi, 0) = f(\xi)$ on $s = 0$, and so

$$u(\xi, s) = 1 - \cos s + f(\xi)$$

Inverting variables: $s = t$ and $\xi = xe^{-s} = xe^{-t}$. So solution is, in terms of (x, t)

$$u(x, t) = 1 - \cos t + f(xe^{-t})$$

In this example, the characteristic curves are not straight lines – given by $\xi = xe^{-t} = \text{constant}$. Note also, solution is not a constant along the characteristic curves.



6.1.3 Semi-linear PDEs

These have a general form $a(x, t)u_t + b(x, t)u_x = c(x, t, u)$. Can be non-linear, but must be linear in the derivatives.

E.g.: Consider

$$u_t + cu_x = u^2, \quad \text{with } u(x, 0) = \cos x$$

Same method as before... so the characteristic equations are:

$$(i): \frac{dt}{ds} = 1, \quad (ii): \frac{dx}{ds} = c, \quad (iii): \frac{du}{ds} = u^2$$

Integrating up and imposing $x = \xi, s = 0$ on $t = 0$ gives

$$(i): t = s, \quad (ii): x = cs + \xi, \quad \text{and } (iii): \frac{-1}{u} = s + C(\xi)$$

where $C(\xi)$ is to be determined by the transformed IC: $u(\xi, 0) = \cos \xi$ when $s = 0$. So easily find

$$u(\xi, s) = \frac{\cos \xi}{1 - s \cos \xi}$$

and inverting variables: $s = t$ and $\xi = x - ct$ (the characteristics are straight lines) gives

$$u(x, t) = \frac{\cos(x - ct)}{1 - t \cos(x - ct)}$$

Note: solution 'blows up' when $t = 1$ and $\cos(x - c) = 1$... the strange+interesting effects of **non-linearity**.

6.1.4 Quasi-linear PDEs

Consider the PDE of a general type

$$u_t + g(u)u_x = 0, \quad \text{with I.C. } u(x, 0) = f(x)$$

where g is a given function of one variable. Must be linear in u_t and u_x , but may be nonlinear in u .

Same method again... so the characteristic equations are:

$$(i): \frac{dt}{ds} = 1, \quad (ii): \frac{dx}{ds} = g(u), \quad (iii): \frac{du}{ds} = 0$$

Integrating up and imposing $x = \xi, s = 0$ on $t = 0$ gives

$$(i): t = s, \quad \text{and} \quad (iii): u = C(\xi)$$

where $C(\xi)$ is to be determined by the transformed IC: $u(\xi, 0) = f(\xi)$ when $s = 0$. I.e. $u(\xi, s) = f(\xi)$.

We have missed out (ii), but can now integrate that too since

$$\frac{dx}{ds} = g(f(\xi)), \quad \Rightarrow \quad x = sg(f(\xi)) + \xi$$

Using $s = t$, we have

$$\boxed{x = tg(f(\xi)) + \xi} \quad (6.2)$$

as the equation defining characteristic curves. These are **straight** lines in the (x, t) plane with slope $1/g(f(\xi))$. Solution is constant along characteristics, with

$$u(x, t) = f(\xi) = f(x - tg(f(\xi))).$$

Note: Since $u = f(\xi)$, can eliminate ξ :

$$u(x, t) = f(x - tg(u))$$

which is an *implicit* equation defining $u(x, t)$.

6.2 Examples of Nonlinear Wave Problems

6.2.1 Examples with Shock Waves

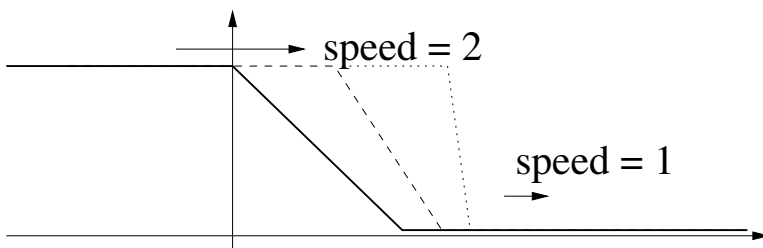
Example 1. Suppose

$$g(u) = 1 + u, \quad \text{so} \quad u_t + (1 + u)u_x = 0. \quad (6.3)$$

with

$$u(x, 0) = f(x) = \begin{cases} 1 & \text{for } x \leq 0 \\ 1 - x & \text{for } 0 < x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$

A simple physical interpretation of the solution goes as follows. The wave speed $c = 1 + u = 1 + \text{initial height}$. So for $x < 0$, $c = 1 + 1 = 2$ and travels faster than for $x > 1$, where $c = 1 + 0 = 1$.

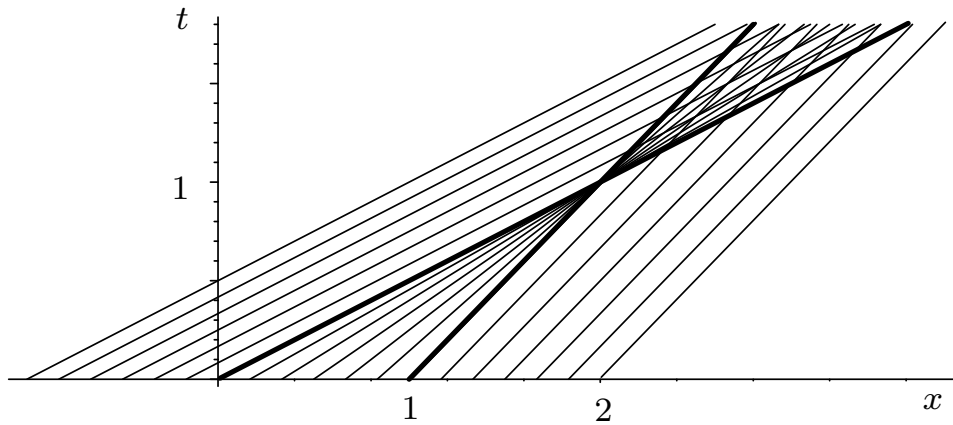


In the I.C. we can interchange x with ξ .

So

- For $\xi < 0$, $g(f(\xi)) = 1 + u(\xi, 0) = 1 + 1 = 2$. So slope of char. is $1/2$ and equation of char. is $x = 2t + \xi$ (from (6.2)).
- For $\xi \geq 1$, $g(f(\xi)) = 1 + u(\xi, 0) = 1 + 0 = 1$. So slope of char is 1 and equation of char. is $x = t + \xi$
- For $0 < \xi < 1$, $g(f(\xi)) = 1 + 1 - \xi = 2 - \xi$. So slope of char is $1/(2 - \xi)$ and equation of char is $x = (2 - \xi)t + \xi$.

Sketch of chars:



- The two bounding curves are (A): $x = 2t$ and (B): $x = t + 1$. They cross when (x, t) are the same, i.e. at $1 + t = 2t$ or $t = 1$, $x = 2$.
- Interior curves are given by $x = (2 - \xi)t + \xi$ for $0 < \xi < 1$. Easy to see they all pass through $x = 2$, $t = 1$.

Solution: $u = f(\xi)$ so

$$u(x, t) = \begin{cases} 1 & \text{for } \xi \leq 0 \\ 0 & \text{for } \xi \geq 1 \\ 1 - \xi & \text{for } 0 < \xi < 1 \end{cases}$$

or, in terms of (x, t) ,

$$u(x, t) = \begin{cases} 1 & \text{for } x \leq 2t \\ 0 & \text{for } x \geq 1 + t \\ \left(\frac{1 + t - x}{1 - t} \right) & \text{for } 2t < x < 1 + t \end{cases}$$

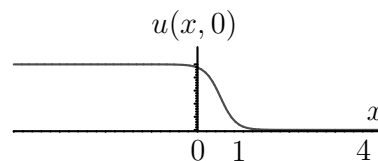
(last equation comes from $x = (2 - \xi)t + \xi$ implies $\xi(1 - t) = x - 2t$ implies $\xi = (x - 2t)/(1 - t)$ and so $1 - \xi = 1 - ((x - 2t)/(1 - t))$).

Can see that the solution **blows up** at $t = 1$. This is where the characteristics cross one another. Always the case: *when characteristics cross the solution breaks down*. Indicative of **shock waves**.

Physically, in this problem, it is where the linear ramp becomes vertical; infinite gradient implies derivatives don't exist.

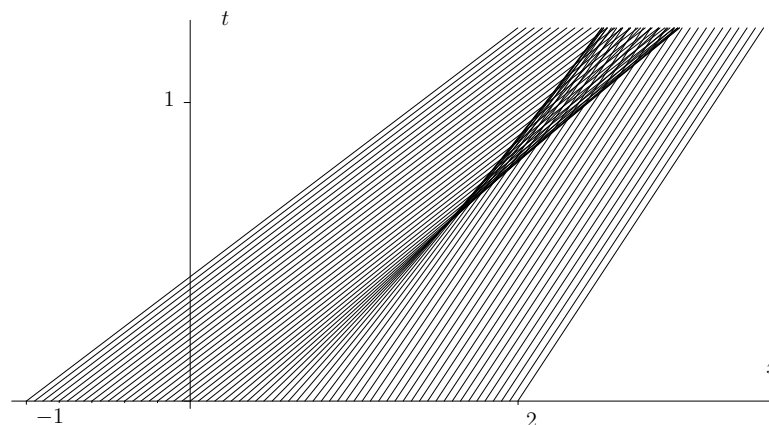
Very interesting... but not here.

Example 2. The same PDE but a smooth initial condition: $\frac{1}{2}[1 - \tanh 3(x - \frac{1}{2})]$. Here is its graph:



It is a smoothed-out version of Example 1.

Characteristics:

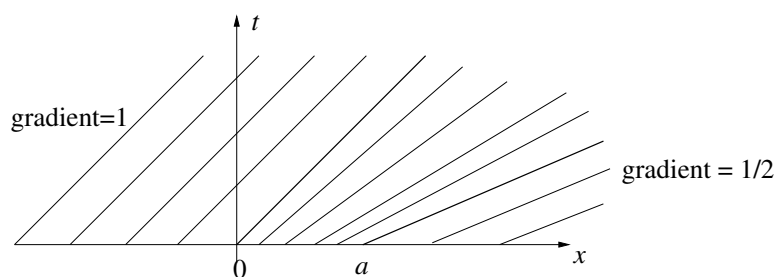


6.2.2 Example with an Expansion Fan

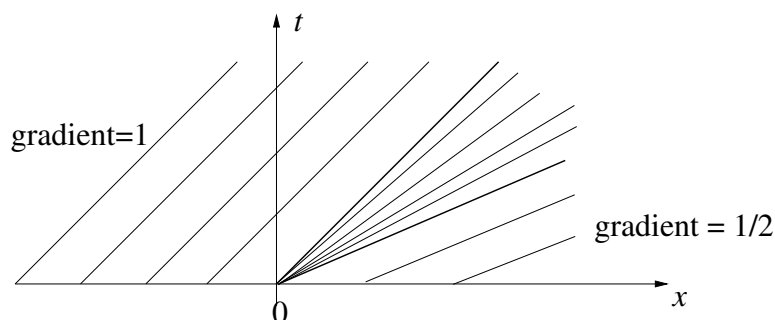
Example 3: The same PDE $u_t + (1 + u)u_x = 0$, but this time with an initial condition which increases with x :

$$u(x, 0) = f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x/a & \text{for } 0 < x < a \\ 1 & \text{for } x \geq a \end{cases} \quad \text{where } a > 0$$

The characteristics spread out, and there are no shocks in this problem.



In the limit as $a \rightarrow 0$ the solution is called an *expansion fan*.



6.3 Traffic Flow

This theory was invented in Manchester in 1955 by Sir James Lighthill and G. B. Whitham.

6.3.1 The Traffic Flow Equation

Let x be distance along a road (not necessarily straight).

Traffic density $\rho(x, t)$ on a road is defined as the number of cars (or other vehicles) per unit distance at the point x and time t .

Then the number of cars at time t in the region $a < x < b$ is $\int_a^b \rho(x, t) dx$.

ρ is really a subtle kind of average.

Conservation Law: No cars can be created or destroyed and so can use the conservation law in §2.3.1 ('stuff' is now cars).

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \phi}{\partial x}$$

where $\phi(x, t)$ is **flux**. In this context, flux is *the rate at which cars are crossing the fixed point x* . I.e. it is (density of cars) $\times$ (speed of cars).

$$\phi(x, t) = \rho(x, t)u(x, t)$$

where $u(x, t)$ is the **Traffic speed**. So we have

$$0 = \rho_t + (\rho u)_x = \rho_t + \rho_x u + \rho u_x \quad (6.4)$$

This **one** equation involves **two** dependent variables ρ and u . Need another equation linking ρ and u to close the model.

Instead of $u = u(x, t)$, propose $u = u(\rho) = u(\rho(x, t))$.

I.e. ones speed is not dependent where you are on the road, or what time it is, only on the density of the traffic surrounding you. So

$$\phi = \rho u = \rho u(\rho) \equiv f(\rho), \quad \text{say.}$$

Now we get

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} f(\rho) = \rho_t + f'(\rho) \rho_x \quad (6.5)$$

This is a quasi-linear PDE of the type already investigated in §6.1.4.

6.3.2 The Quadratic Model

Assumptions:

1. When $\rho = 0$, $u = u_{max}$, the maximum speed a car will travel at on an empty road (the speed limit ?)
2. $u = 0$ when $\rho = \rho_c$ where $\rho_c = 1/\text{spacing between cars in a jam}$.

Consider the simplest model, in which u is a linear function of ρ :

$$u = u_{max} \left(1 - \frac{\rho}{\rho_c} \right) \quad \text{for } \rho \leq \rho_c \quad (6.6)$$

Now $f(\rho) = \rho u = u_{max} \rho (1 - \rho/\rho_c)$ and

Plug into the conservation eqn (6.5) to get

$$\boxed{\rho_t + c(\rho_c - 2\rho)\rho_x = 0} \quad (6.7)$$

where

$$c = \frac{u_{max}}{\rho_c}. \quad (6.8)$$

This is the type of PDE considered in §6.2, but with a negative coefficient for the quadratic term $\rho\rho_x$.

Wave speed = $1/(\text{slope of characteristics}) = c(\rho_c - 2\rho)$. Can be +ve or -ve.

Speed of traffic is $c[\rho_c - \rho] > 0$ from (6.6) and (6.8).

So the *wave speed* < *traffic speed*.

This means that changes in density travel more slowly than cars. So when you drive, you go faster than the changes in density; that's why you have to slow down to avoid

thickening of traffic. This is the other way round from water waves, where as you float with the wave, breakers come up behind you. The reason is the nonlinear term $\rho\rho_x$ in (6.7) has a minus sign, where the nonlinear term in (6.3) has a plus sign.

It is easy to verify that if $\rho(x, 0)$ is an increasing function, you get shock formation, so when traffic starts to get thicker (as when a motorway lane closes), discontinuities tend to develop. Anyone who has driven on a busy motorway knows that traffic jams can form suddenly and for no obvious reason. They are shock waves formed by the steepening of initially smooth changes of density.

Remark:

These quasi-linear equations also closely connected to other observable phenomena, such as glacier flows and sedimentation in river deltas.

Chapter 7

PDEs in $\mathbb{R}^2$ and $\mathbb{R}^3$

7.1 Harmonic Functions

In this section we investigate a very special class of functions called *harmonic*. We will be concentrating on harmonic functions in $\mathbb{R}^2$ but the results of this section are valid in $\mathbb{R}^n$ and most proofs are transferrable directly to $\mathbb{R}^n$.

DEFINITION 7.1.1 Let $n \in \mathbb{N}^+$ and $\Omega \subset \mathbb{R}^n$ be an open connected set. A function $u \in C^2(\Omega)$ is called **harmonic** if $\Delta u(x) = 0$ for all $x \in \Omega$.

The basic examples of harmonic functions are

EXAMPLE 7.1.2 Here are several examples of harmonic functions in $\mathbb{R}^2$:

1. $u(x, y) = x + y$, $\Omega \subset \mathbb{R}^2$ is any open set;
2. $u(x, y) = x^2 - y^2$, $\Omega \subset \mathbb{R}^2$ is any open set;
3. $u(x, y) = \ln(x^2 + y^2)$, $\Omega \subset \mathbb{R}^2$ is any open set that does not include 0.

Note that in the third example if Ω contains 0 then $u(x, y)$ will not be harmonic as it is not in $C^2(\Omega)$.

7.1.1 Mean Value Property

Harmonic functions have many very nice properties. Here we prove that harmonic functions satisfy *the mean value property (MVP)*. We always denote by $B_r(x)$ a ball of radius r centered at a point x .

DEFINITION 7.1.3 Let $\Omega \subset \mathbb{R}^2$ be an open connected set and $u \in C(\Omega)$. We say that

1. u satisfies the **first mean value property** in Ω if

$$u(x) = \frac{1}{2\pi r} \int_{\partial B_r(x)} u(y) dS_y, \quad \text{for any } B_r(x) \subset \Omega.$$

Note that we can rewrite the above as

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta, \quad (7.1)$$

where $\hat{n}(\theta) = (\cos \theta, \sin \theta)$, $r > 0$ is such that $B_r(x) \subset \Omega$.

2. u satisfies the **second mean value property** in Ω if

$$u(x) = \frac{1}{\pi r^2} \int_{B_r(x)} u(y) dy, \quad \text{for any } B_r(x) \subset \Omega.$$

Note that we can rewrite the above as

$$u(x) = \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r u(x + s\hat{n}(\theta)) s d\theta, \quad (7.2)$$

where $\hat{n}(\theta) = (\cos \theta, \sin \theta)$, $r > 0$ is such that $B_r(x) \subset \Omega$.

Now we show that the first and the second MVPs are equivalent and therefore later we refer to both of them as mean value property.

PROPOSITION 7.1.4 *A function $u \in C(\Omega)$ satisfies the first mean value property if and only if it satisfies the second mean value property.*

Proof. Assume u satisfies the first mean value property. Take any $r > 0$ such that $B_r(x) \subset \Omega$, then for all $0 < s < r$ we have

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + s\hat{n}(\theta)) d\theta.$$

Multiplying both parts by $2\pi s$ and integrating from 0 to r we obtain

$$\pi r^2 u(x) = \int_0^{2\pi} \int_0^r u(x + s\hat{n}(\theta)) s d\theta.$$

Hence we obtain the second mean value property.

Assume now that the second mean value property is satisfied. Take any $r > 0$ such that $B_r(x) \subset \Omega$, then we have

$$\pi r^2 u(x) = \int_0^{2\pi} \int_0^r u(x + s\hat{n}(\theta)) s d\theta.$$

Differentiating with respect to r we obtain

$$2\pi r u(x) = \int_0^{2\pi} u(x + r\hat{n}(\theta)) r d\theta.$$

Dividing both parts by $2\pi r$ we obtain the result. $\square$

Now we would like to show that functions satisfying mean value property and harmonic functions are indeed the same. We first proof the following result.

PROPOSITION 7.1.5 Let $\Omega \subset \mathbb{R}^2$ be open connected domain and $u \in C^2(\Omega)$ be a harmonic function. Then u satisfies mean value property in Ω .

Proof. Let $x \in \Omega$ be any point. We take any $r > 0$ such that $B_r(x) \subset \Omega$ and want to show that if $\Delta u = 0$ in Ω then

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta.$$

In order to prove this result we will use integration by parts or Green's identity. The divergence (or Gauss) theorem tells us that for any vector field $F(x)$ defined for $x \in \Omega$ and any $\Omega_0 \subset \Omega$ we have

$$\int_{\Omega_0} \operatorname{div} F(x) dx = \int_{\partial\Omega_0} F(x) \cdot \hat{n} dS(x),$$

where n is a normal vector to Ω_0 . Now we take $\Omega_0 = B_r(0) \setminus B_\epsilon(0)$. Since $\Delta u = 0$ and $\Delta \ln |x| = 0$ for $x \neq 0$ we have the following equality

$$\begin{aligned} 0 &= \int_{\Omega_0} u(x+y) \Delta \ln |y| - \ln |y| \Delta u(x+y) dy = \int_{\Omega_0} \operatorname{div}(u(x+y) \nabla \ln |y| - \ln |y| \nabla u(x+y)) dy \\ &= \int_{\partial B_r(0)} u(x+y) \nabla \ln |y| \cdot \hat{n} dS(y) - \int_{\partial B_\epsilon(0)} u(x+y) \nabla \ln |y| \cdot \hat{n} dS(y) + \\ &\quad \int_{\partial B_r(0)} \ln |y| \nabla u(x+y) dS(y) - \int_{\partial B_\epsilon(0)} \ln |y| \nabla u(x+y) dS(y) \quad (7.3) \end{aligned}$$

We can simplify the right hand side to obtain

$$\int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta - \int_0^{2\pi} u(x + \epsilon\hat{n}(\theta)) d\theta = 0.$$

Here we used the fact that

$$\int_{\partial B_r(0)} \ln |y| \nabla u(x+y) dS(y) = \ln |y| \int_{\partial B_r(0)} \nabla u(x+y) dS(y) = -\ln |y| \int_{\partial B_r(0)} \Delta u(x+y) dy = 0.$$

Now we have

$$\int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta = \int_0^{2\pi} u(x + \epsilon\hat{n}(\theta)) d\theta$$

and taking $\epsilon \rightarrow 0$ we obtain

$$\int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta = 2\pi u(x).$$

So we prove the result. $\square$

We can also give a different proof of this result.

Proof 2. Let $x \in \Omega$ be any point. We take any $r > 0$ such that $B_r(x) \subset \Omega$ and want to show that if $\Delta u = 0$ in Ω then

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta.$$

We are using integration by parts

$$\begin{aligned} 0 &= \int_{B_r(x)} \Delta u(y) dy = \int_{B_r(0)} \Delta u(x+y) dy = \int_{\partial B_r(0)} \nabla u(x+y) \cdot \hat{n} dS(y) \\ &= r \int_0^{2\pi} \frac{\partial}{\partial r} u(x + r\hat{n}(\theta)) d\theta = r \frac{\partial}{\partial r} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta. \quad (7.4) \end{aligned}$$

Therefore we have

$$\int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta = \int_0^{2\pi} u(x) d\theta = 2\pi u(x).$$

The result follows. $\square$

Now we understand that harmonic functions satisfy mean value property and want to prove the opposite result.

PROPOSITION 7.1.6 *Let $\Omega \subset \mathbb{R}^2$ be open connected domain and $u \in C^2(\Omega)$ satisfies mean value property in Ω . Then u is a harmonic function.*

Proof. Let $x \in \Omega$ be any point. Want to show that if

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta$$

for all $B_r(x) \subset \Omega$ then $\Delta u(x) = 0$. Using formula (7.4) we see that for all $B_r(x)$

$$\begin{aligned} \int_{B_r(x)} \Delta u(y) dy &= \int_{B_r(0)} \Delta u(x + y) dy = \int_{\partial B_r(0)} \nabla u(x + y) \cdot \hat{n} dS(y) \\ &= r \int_0^{2\pi} \frac{\partial}{\partial r} u(x + r\hat{n}(\theta)) d\theta = r \frac{\partial}{\partial r} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta = 2\pi r \frac{\partial}{\partial r} u(x) = 0. \end{aligned} \quad (7.5)$$

Now we take an average and let $r \rightarrow 0$ to obtain

$$0 = \lim_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{B_r(x)} \Delta u(y) dy = \Delta u(x).$$

The result is proved. $\square$

7.1.2 Maximum Principle

In this section we prove a maximum principle for harmonic functions. We start with the following result.

PROPOSITION 7.1.7 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a harmonic function defined on Ω . Assume u achieves maximum at a point $x_0 \in \Omega$ then $u(x) \equiv \text{const}$ for all $x \in \Omega$.*

Proof. Since $x_0 \in \Omega$ and Ω is an open set we can find $r > 0$ such that $B_r(x_0) \subset \Omega$ and therefore by mean value property we have

$$u(x_0) = \frac{1}{\pi r^2} \int_{B_r(x_0)} u(x) dx.$$

Since $u(x_0) \geq u(x)$ for all $x \in \Omega$ the only way to satisfy mean value property is to have $u(x) = u(x_0)$ for all $x \in B_r(x_0)$. Now take any point $x_n \in \Omega$, we want to show that $u(x_n) = u(x_0)$. We can connect x_0 and x_n by a continuous curve that we cover by intersecting balls $B_{r_0}(x_i) \subset \Omega$ with $2r_0 < r$ in such a way that $|x_{i+1} - x_i| < r_0$ for

$i = 0, \dots, n - 1$. By the first step we already know that $u(x_1) = u(x_0)$, repeating the arguments we obtain that

$$u(x_0) = u(x_i), \quad \text{for all } i = 1, \dots, n.$$

Therefore we have the result. $\square$

Changing u to $-u$ we obtain the following result.

COROLLARY 7.1.8 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a harmonic function defined on Ω . Assume u achieves minimum at a point $x_0 \in \Omega$ then $u(x) \equiv \text{const}$ for all $x \in \Omega$.*

Note that we only used mean value property to prove maximum principle. Using above results we clearly have

PROPOSITION 7.1.9 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a harmonic function defined on Ω . Then u attains its minimum and maximum values over $\bar{\Omega}$ on $\partial\Omega$.*

Now we want to use maximum principle to show some uniqueness properties of harmonic functions.

PROPOSITION 7.1.10 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a harmonic function defined on Ω . Then u is uniquely defined by its values on the boundary $\partial\Omega$.*

Proof. Assume there is a harmonic function v defined on $\bar{\Omega}$ and such that $v(x) = u(x)$ for all $x \in \partial\Omega$. It is clear that a difference $w(x) = u(x) - v(x)$ is also a harmonic function. Moreover, $w(x) = 0$ for all $x \in \partial\Omega$. Since we know that minimum and maximum of harmonic function are achieved at the boundary by obtain $0 \leq w(x) \leq 0$ for all $x \in \bar{\Omega}$. Therefore $w(x) \equiv 0$ and $v(x)$ coincides with $u(x)$. $\square$

COROLLARY 7.1.11 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a solution of a Poisson equation*

$$\Delta u(x) = f(x) \quad \text{for all } x \in \Omega, \quad u(x) = u_0(x) \quad \text{for all } x \in \partial\Omega.$$

Then u is unique solution.

Proof. Assume there are two solutions u and v . Then their difference $w(x) = u(x) - v(x)$ is harmonic and zero on $\partial\Omega$. Therefore $w(x) \equiv 0$ and we have a contradiction. $\square$

7.2 The Laplacian in non-Cartesian Coordinates

7.2.1 2D Polars (plane polars)

We transform $\nabla^2 u = u_{xx} + u_{yy}$ to (r, θ) coordinates, where

$$\left. \begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta \end{aligned} \right\},$$

Application of the chain rule eventually gives:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \quad (7.6)$$

7.2.2 3D: Cylindrical Polar Coordinates

Cylindrical polar coordinates are (r, θ, z) with $x = r \cos \theta$, $y = r \sin \theta$ as before, Then

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \quad (7.7)$$

7.3 Separation solutions

7.3.1 Cartesian Coordinates (2D)

Consider $\nabla^2 u = 0$ inside a rectangular domain, $0 < x < a$, $0 < y < b$, say. Then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $u(x, y) = X(x)Y(y)$. Then $X''(x)Y(y) + X(x)Y''(y) = 0$ and so

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k$$

where k is the separation constant. Need B.C's to determine k .

Example: if $u(0, y) = 0$, $u(a, y) = 0$ then $k = -\mu^2$ and $X(x) = \sin(n\pi x/a)$, where $\mu = n\pi/a$.

Then solving for $Y(y)$, ($Y''(y) = \mu^2 Y(y)$) gives

$$Y(y) = A_n \sinh(n\pi y/a) + B_n \cosh(n\pi y/b)$$

or

$$Y(y) = C_n e^{(n\pi y/a)} + D_n e^{-(n\pi y/a)}$$

(typical to use the former representation if the y -domain is finite, latter if infinite).

E.g. 1 Let $u(x, 0) = 0$ and $u(x, b) = f(x)$. Then

$$u(x, y) = \sum_{n=1}^{\infty} (A_n \sinh(n\pi y/a) + B_n \cosh(n\pi y/a)) \sin(n\pi x/a)$$

So $u(x, 0) = 0$ implies $B_n = 0$ for all n and $u(x, b) = f(x)$ implies

$$f(x) = \sum_{n=1}^{\infty} (A_n \sinh(n\pi b/a)) \sin(n\pi x/a)$$

and then, using expansion formula,

$$A_n \sinh(n\pi b/a) = \frac{\langle f, \sin(n\pi x/a) \rangle}{\| \sin(n\pi x/a) \|^2}$$

determines A_n and hence u .

E.g. 2 if $u(x, 0) = f(x)$ and $u(x, y) \rightarrow 0$ as $y \rightarrow \infty$. For $0 < x < a$ then $C_n = 0$ in above (for bounded solutions) and

$$u(x, y) = \sum_{n=1}^{\infty} D_n e^{(-n\pi y/a)} \sin(n\pi x/a)$$

is general solution. Find D_n by putting $y = 0$ with

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} D_n \sin(n\pi x/a)$$

and continue as in E.g. 1.

Of course, D_n (and previously $A_n \sinh(n\pi b/a)$) are the coefficients of the Fourier Sine Series for $f(x)$ (see section 3).

7.3.2 Plane Polars

If a 2D problem has boundaries which fit naturally to a circular geometry then separation in polars is natural.

For example, solving $\nabla^2 u = 0$ inside a circle, $r < a$, with $u(r, \theta) = f(\theta)$ on $r = a$.

I.e. Solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

Separation ? Look for solutions of the form

$$u(r, \theta) = R(r)\Theta(\theta)$$

Plug in

$$\Theta R'' + \frac{\Theta R'}{r} + \frac{R \Theta''}{r^2} = 0$$

and divide by $R\Theta/r^2$

$$\frac{r^2 R''}{R} + \frac{r R'}{R} = -\frac{\Theta''}{\Theta} = k$$

where k is a separation constant.

So we have

$$r^2 R'' + r R' - k R = 0, \quad \text{and} \quad \Theta'' + k \Theta = 0. \quad (7.8)$$

7.3.3 The Θ Equation

To find the separation constant, we want an inhomog. equation with inhomog BC's. The $R(r)$ -eqn won't do it, but the $\Theta(\theta)$ -eqn will...

General solutions are

$$\Theta = A \cos(\sqrt{k} \theta) + B \sin(\sqrt{k} \theta) \quad (7.9)$$

Note that if $k < 0$ then $\cos$ and $\sin$ become $\cosh$ and $\sinh$.

On our original problem, we assume u and its derivatives are continuous for all r, θ . So we must insist that $u(r, \theta) = u(r, \theta + 2\pi)$ and $u_\theta(r, \theta) = u_\theta(r, \theta + 2\pi)$.

Looking at (7.9) we can do this if $\sqrt{k} = m$ (or $k = m^2$) where m is an integer. Then

$$\Theta = A_m \cos(m\theta) + B_m \sin(m\theta), \quad m \in \mathbb{Z}$$

Notes:

- Only need $m \geq 0$ since $m < 0$ gives the same functions with B_m replaced with $-B_m$.
- with $k < 0$ $\cosh$ and $\sinh$ functions won't work.

7.3.4 The R Equation

The R equation in (7.8) with $k = m^2$ gives

$$r^2 R'' + r R' - m^2 R = 0 \quad (7.10)$$

Solution ? Note non-constant coefficients. Try $R(r) = r^\alpha$ where α is a constant. Then (7.10) is

$$\begin{aligned} \alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - m^2 r^\alpha &= 0 \\ \implies (\alpha^2 - m^2)r^\alpha &= 0, \end{aligned}$$

so $\alpha^2 = m^2$, and $\alpha = \pm m$.

General solution is

$$R(r) = C_m r^m + D_m / r^m \quad (7.11)$$

However, when $m = 0$, r^m and r^{-m} are the same functions – 1, so there must be another...

The equation for $m = 0$ is $r^2 R'' + r R' = 0$. Easy to solve.

For $r \neq 0$ we have $r \frac{dR'}{dr} = -R'$. Separate variables and integrate to get $\log R = -\log r + \log D_0$ so that $R'(r) = D_0/r$. Then integrate again to get R , giving

$$R(r) = C_0 + D_0 \log r \quad (7.12)$$

7.3.5 The full solution

Putting all different solutions together using superposition gives the general solution

$$u(r, \theta) = (C_0 + D_0 \log r)A_0 + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m} \right) (A_m \cos m\theta + B_m \sin m\theta) \quad (7.13)$$

[Note: For $m = 0$ we have $A_0 \cos 0\theta + B_0 \sin 0\theta = A_0$, giving $u(r, \theta) = A_0(C_0 + D_0 \log r)$.]

Example 1. $\nabla^2 u = 0$ for $r < a$ with BC $u(a, \theta) = f(\theta)$ where f is a given function for $0 < \theta < 2\pi$.

The domain includes the point $r = 0$. Must avoid singularities (infinities) in the solution and so $D_m = 0$ for all m and $D_0 = 0$.

Hence

$$u(r, \theta) = a_0 + \sum_{m=1}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta)$$

where $a_m = A_m C_m$ and $b_m = B_m C_m$ in the notation of (7.13).

The B.C. at $r = a$ gives

$$a_0 + \sum_{m=1}^{\infty} a^m (a_m \cos m\theta + b_m \sin m\theta) = f(\theta) \quad \text{for } 0 < \theta < 2\pi$$

where f is a given function. Thus a_m, b_m are (apart from factors of a^m) the Fourier Series coefficients. Find using expansion formula (section 3).

Example 2. $\nabla^2 u = 0$ in $1 < r < 2$. This region is called an **annulus**.

B.C.'s needed on $r = 1, 2$:

$$u(1, \theta) = f(\theta), \quad u(2, \theta) = g(\theta) \quad \text{for } 0 < \theta < 2\pi$$

where f and g are given functions.

The solution is given by (7.13), but can include all the D_m 's and D_0 as $r = 0$ is not part of the annular region. Follow as before but apply conditions on both $r = 1$ and $r = 2$ and get coupled equations for C_n and D_n .

7.4 Diffusion in a Cylinder

Consider diffusion in a long cylinder (e.g. heat flow in a hot water pipe). Choose cylindrical polars, z along cylinder axis.

Assume u is independent of z and θ . So $u = u(r, t)$ and satisfies

$$u_t = D \left(u_{rr} + \frac{1}{r} u_r \right) \left(\equiv D \frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) \right) \quad (7.14)$$

where $D > 0$ is the diffusion coefficient.

We need an initial condition:

$$u(r, 0) = f(r) \text{ for } 0 < r < a \quad (7.15)$$

We also need a B.C. on $r = a$, so consider

$$u(a, t) = 0 \text{ for } t > 0 \quad (7.16)$$

Separation of Variables

Let $u(r, t) = R(r)T(t)$, substitute into (7.25):

$$\frac{T'}{DT} = \frac{1}{R} \left(R'' + \frac{R'}{r} \right) = -k.$$

(We chose $-k$, because from what we know about diffusion we expect exponential decay in time, thus implying that $k > 0$ in the above assignment)

The T eqn: Easy $T' = -kDT$ has solutions Ce^{-kDt} . Still need to know what values k takes.

The R eqn: is

$$(rR')' + krR = 0. \quad (7.17)$$

This is the same as (7.20) with $m = 0$ and $k = \lambda$. So solutions bounded at $r = 0$ given by Bessel functions $J_0(r\sqrt{k})$ and

$$R(r) = BJ_0(r\sqrt{k}) \quad (7.18)$$

for constant B .

Values of k determined by B.C. $R(a) = 0$. I.e. $J_0(a\sqrt{k}) = 0$ so $\sqrt{k} = z_{0,i}/a$, $i = 1, 2, \dots$ and

$$R(r) = B_i J_0(z_{0,i}r/a)$$

are the eigenfunctions.

General solution

Superposition of all sep. solutions gives a general solution

$$u(r, t) = \sum_{i=1}^{\infty} a_i e^{\left(\frac{-z_{0,i}^2 D t}{a^2} \right)} J_0 \left(\frac{z_{0,i} r}{a} \right) \quad (7.19)$$

for unknown coefficients a_i , which are contracted from C and B_i .

To find a_i , apply the I.C. $u(r, 0) = f(r)$, $r < a$ so

$$\sum_{i=1}^{\infty} a_i J_0(z_{0,i}r/a) = f(r) \quad \text{for } 0 < r < a \quad (7.20)$$

From the expansion theorem (this is all S-L),

$$a_i = \frac{\langle f(r), J_0(z_{0,i}r/a) \rangle}{\langle J_0(z_{0,i}r/a), J_0(z_{0,i}r/a) \rangle} \equiv \frac{\int_0^a f(r) J_0(z_{0,i}r/a) r dr}{\int_0^a J_0^2(z_{0,i}r/a) r dr}$$

which can be found (at least numerically).

Note that the orthogonality result of Bessel functions is, ensured by SL theory is

$$\langle J_0(z_{0,i}r/a), J_0(z_{0,j}r/a) \rangle \equiv \int_0^a J_0(z_{0,i}r/a) J_0(z_{0,j}r/a) r dr = 0, \quad i \neq j$$

7.5 The Wave Equation: Normal Modes

7.5.1 Normal modes for the 2D wave equation

Consider

$$u_{tt} = c^2 \nabla^2 u \equiv c^2 (u_{xx} + u_{yy}) \quad (7.21)$$

inside a domain $\mathcal{D}$.

Solution determined by:

- Initial values of u and u_t at all points of $\mathcal{D}$,
- Values of u on S , boundary of $\mathcal{D}$ for all t .

We shall only consider the case where $u = 0$. This corresponds to vibrations on a drum skin with fixed edges. Easy to generalise to setting the normal derivative of u equal to zero on S .

The simplest vibration is sinusoidal in time. I.e. motion is proportional to $\sin \omega t$ or $\cos \omega t$ where **period** of oscillations is $2\pi/\omega$.

A **normal-mode solution** of (7.14) to be a solution of the form

$$u(x, y, t) = \phi(x, y) \cos(\omega t + \delta) \quad (7.22)$$

where δ is constant phase-shift.

Plugging (7.15) into (7.14) gives

$$\begin{aligned} -\omega^2 \phi(x, y) \cos(\omega t + \delta) &= c^2 (\nabla^2 \phi) \cos(\omega t + \delta) \\ \implies -\nabla^2 \phi &= (\omega^2 / c^2) \phi \end{aligned}$$

The function ϕ is an eigenfunction of $-\nabla^2$ with eigenvalue $\lambda = \omega^2 / c^2$. So the angular frequency is

$$\omega = c\sqrt{\lambda}$$

in terms of the eigenvalue.

7.5.2 An Example

We find the eigenvalues and eigenfunctions of $-\nabla^2$ on rectangle $0 < x < a, 0 < y < b$ with $u = 0$ on the boundary.

I.e. solve

$$-(\phi_{xx} + \phi_{yy}) = \lambda\phi \quad (7.23)$$

with $\phi(0, y) = \phi(a, y) = \phi(x, 0) = \phi(x, b) = 0$ Solve (7.16) by separating variables. I.e. let $\phi(x, y) = X(x)Y(y)$, with $X(0) = X(a) = 0$ and $Y(0) = Y(b) = 0$. Then, substitute into (7.16), and the usual argument gives

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda = k$$

where k is a separation constant.

Solve for $X(x)$, so that $k = -n^2\pi^2/a^2$ and $X(x) = \sin(n\pi x/a)$.

The $Y(y)$ equation is then

$$Y''(y) + (k + \lambda)Y(y) = 0$$

Since λ is unknown, we let $k + \lambda = \mu$ so that the Y -eqn is $Y'' + \mu Y = 0$ with $Y(0) = Y(b) = 0$. Just like the eqn for X , we have $\mu = m^2\pi^2/b^2$ with $Y(y) = \sin(m\pi y/b)$ for $m = 1, 2, \dots$ and so $k + \lambda = m^2\pi^2/b^2$ implying:

$$\lambda = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}, \quad n, m = 1, 2, \dots$$

and $\phi(x, y) = \sin(n\pi x/a) \sin(m\pi y/b)$. The frequencies of the normal modes are $\omega = c\sqrt{\lambda}$ so that

$$\omega = \omega_{nm} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

There exist an infinite, discrete set of frequencies.

The shape of the normal mode is constructed from the separate components so that

$$u(x, y, t) = \phi_{nm}(x, y) \cos(\omega_{nm}t + \delta) = \sin(n\pi x/a) \sin(m\pi y/b) \cos(\omega_{nm}t + \delta)$$

Defn The **fundamental frequency** means the lowest value of ω_{nm} which is when $n = m = 1$ and

$$\omega_{11} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

and the corresponding **fundamental mode** is $\phi_{11}(x, y) = \sin(\pi x/a) \sin(\pi y/b)$

EXTRA ONLINE: Square domain

non-examinable content in this section – for interest only

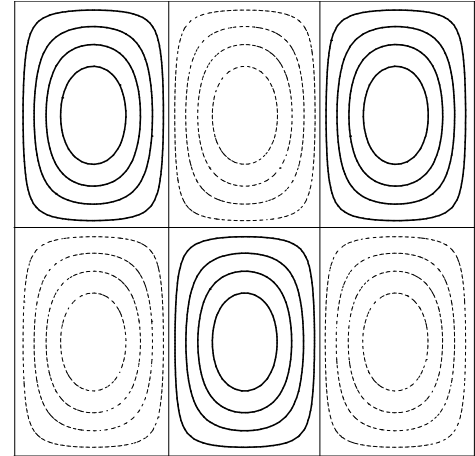
In the simplest case where the domain is a square, with $a = b$, the frequencies ω_{nr} are given by the infinite matrix

$$\omega_{nr} = \frac{c\pi}{a} \begin{pmatrix} \sqrt{2} & \sqrt{5} & \sqrt{10} & \dots \\ \sqrt{5} & \sqrt{8} & \sqrt{13} & \dots \\ \sqrt{10} & \sqrt{13} & \sqrt{18} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad (7.24)$$

The 3,2 mode $\phi_{32}(x, y) = \sin\left(\frac{3\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right)$ is illustrated below by a contour diagram, showing the curves in the x, y plane along which $\phi_{32}(x, y)$ is constant.

The solid contours are where $\phi_{32}(x, y) > 0$ and the dotted contours are where $\phi_{32}(x, y) < 0$. There are three maxima and three minima. The straight lines are where $\phi(x, y) = 0$. They divide the rectangle into six regions, called *cells*; each cell consists of a single peak or valley.

As time increases the peaks and valleys each oscillate up and down with angular frequency $\omega_{32} = (c\pi/a)\sqrt{13}$. When ϕ is increasing in one cell, it is decreasing in the adjacent cells; the peaks become valleys and the valleys become peaks after a time π/ω_{32} .



The other normal modes are similar, but with different numbers of cells in the x and y directions. The fundamental mode has just one cell.

The solution of an initial value problem can be found as a superposition of normal modes. So when you bang a drum, the sound produced is a combination of the normal modes. The principle is similar to Fourier series solutions, but the details are lengthy and beyond the scope of this course.

EXTRA ONLINE: Why the guitar is tuneful and drums are noisy

non-examinable content in this section – for interest only

Guitars

A guitar string satisfies the 1-d wave equation with boundary conditions that $u = 0$ at the endpoints, $x = 0$ and a say. It is easy to see that it has normal modes

$$\sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi ct}{a} + \delta\right), \quad n = 1, 2, \dots$$

The angular frequencies are $c\pi/a$, $2c\pi/a$, $3c\pi/a$, $\dots$; they are integer multiples of the fundamental frequency $c\pi/a$. So the sound wave that travels to your ears is a combination of frequencies which are integer multiples of the fundamental (angular) frequency $c\pi/a$. It

is therefore a periodic function of time with period $2a/c$; the higher frequencies correspond to higher terms in the Fourier series solution of the wave equation.

A periodic sound wave like this is heard by the ear as a musical note. The pitch of the note¹ is determined by the period of the wave; high frequencies give high notes. The Fourier coefficients a_n, b_n determine the character of the sound. If a_1 or b_1 is much larger than all the $n > 1$ coefficients, then the note sounds flute-like and smooth. But if a_n or b_n does not decrease rapidly with n (for example, if the n -th coefficient behaves like $1/n$) then the note sounds quite sharp in character and perhaps even harsh. Thus you can *hear* something about the Fourier coefficients in a musical sound.

Drums

The vertical vibration of a drumskin satisfies the wave equation in 2d. The boundary condition is zero displacement at the edge of the drum. If the drum is rectangular, its vibration is a combination of the normal modes derived above. For a square drum, where $a = b$, the normal modes have frequencies ω given by (7.17); the first few are

$$\omega = \sqrt{2}\pi c/a, \sqrt{5}\pi c/a, \sqrt{8}\pi c/a, \sqrt{10}\pi c/a, \dots$$

They are *not* integer multiples of the fundamental frequency. Therefore the sound produced by a drum is *not* heard as a musical note, it is heard as a noise.

Of course most drums are not square but round. One can work out the normal modes for a circular drum, and the answer shows that their frequencies are not integer multiples of the fundamental frequency. That is why drums bang while strings play tunes.

7.6 The Wave Equation in Plane Polar Coordinates

7.6.1 Separation of Variables

Consider the wave equation in a circular domain (vibrations of a circular drumskin, oscillations on the surface of a cup of tea):

$$u_{tt} = c^2 \nabla^2 u \equiv c^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right], \quad 0 < r < a \quad (7.25)$$

with $u = 0$ on $r = a$.

Let $u(x, y, t) = \phi(r, \theta) \cos(\omega t + \delta)$ as before.

Then

$$-\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) = \left(\frac{\omega}{c} \right)^2 \phi = \lambda \phi$$

and λ is the eigenvalue, to be found.

¹*pitch* describes whether it is a high or a low note

Separate variables: $\phi(r, \theta) = R(r)\Theta(\theta)$ and then above is

$$-\left(R''\Theta + \frac{R'\Theta}{r} + \frac{R\Theta''}{r^2}\right) = \lambda R\Theta \quad (7.26)$$

Divide by $R(r)\Theta(\theta)/r^2$ to get

$$\frac{r^2 R''}{R} + \frac{r R'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = k$$

where k is sep. const.

The Θ Equation

We have $\Theta'' + k\Theta = 0$.

Since we are solving inside a circle, need $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$ and so

$$k = m^2 \quad \text{and} \quad \Theta = A_m \cos m\theta + B_m \sin m\theta, \quad \text{for } m = 0, 1, 2, \dots$$

where A_m, B_m are constants.

The R Equation

With $k = m^2$, so the R equation in (7.19) is

$$R'' + \frac{R'}{r} - \frac{m^2}{r^2} R + \lambda R = 0$$

where λ is unknown eigenvalue (once λ is known then so is ω) and m is an integer.

This equation cannot be solved in terms of elementary functions. But it can be analysed by Sturm-Liouville theory. Instead, put into SL form as

$$(rR')' - \frac{m^2}{r} R + \lambda r R = 0, \quad 0 < r < a \quad (7.27)$$

This is a SL equation with $p(r) = \sigma(r) = r$, $q(r) = -m^2/r$. Must have boundedness of R and R' at $r = 0$ whilst $R(a) = 0$ because u vanishes on the circle $r = a$.

Simplifying the Equation

Rescale the independent variable: $x = r\sqrt{\lambda}$ and let $y(x) = R(x/\sqrt{\lambda})$ or $R(r) = y(r\sqrt{\lambda})$. Then $d/dr = \sqrt{\lambda}d/dx$ so that

$$\begin{aligned} \sqrt{\lambda}(xy')' - \frac{m^2\sqrt{\lambda}}{x}y(x) + \frac{\lambda x}{\sqrt{\lambda}}y(x) &= 0 \\ \implies x^2y''(x) + xy'(x) + (x^2 - m^2)y(x) &= 0 \end{aligned} \quad (7.28)$$

This is called **Bessel's equation**.

7.6.2 Solutions of Bessel's Equation

Bessel's equation (7.21) does not have solutions in terms of elementary functions. Their solutions are called Bessel functions. They are well-studied and have many useful properties.

There are two linearly independent solutions:

$$\begin{cases} J_m(x), & \text{(bounded at the origin } \sim x^m) \\ Y_m(x), & \text{(singular at the origin } \sim x^{-m} \log(x)) \end{cases}$$

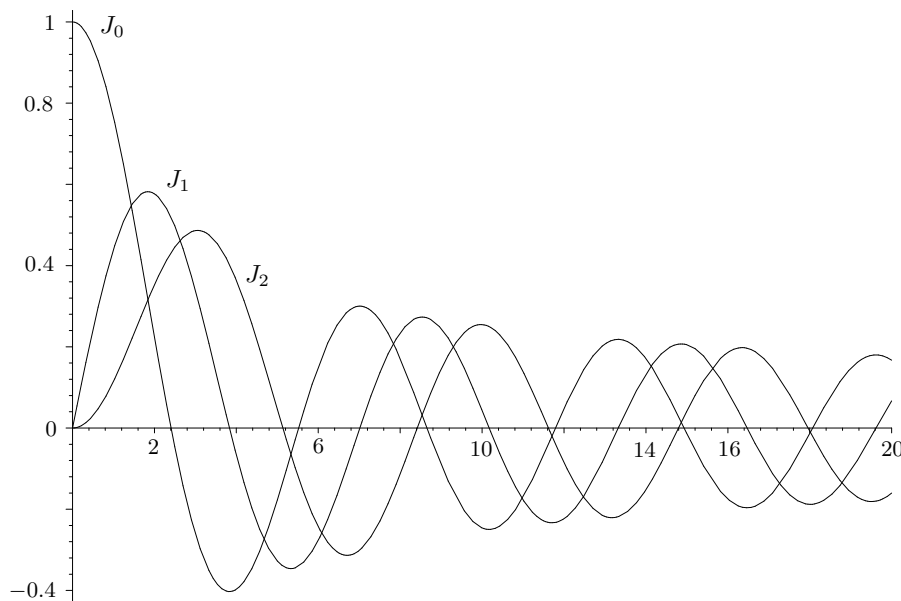
[We won't include Y_m as we don't want to include singularities at $x = 0$ ($r = 0$) in our solution, although for problems which exclude the origin you must include them (not in this course)]

The functions $J_m(x)$ have the following features:

1. Power series representation (cf. cos or sin)

$$J_m(x) = \frac{x^m}{2^m m!} \left[1 - \frac{x^2}{2^2 1!(m+1)} + \frac{x^4}{2^4 2!(m+1)(m+2)} - \dots \right] \quad (7.29)$$

2. Sketch:



The first three Bessel functions.

3. $J_m(x)$ are roughly like $\cos(x + \epsilon)/x^{1/2}$ for large x .
4. $J_0(0) = 1$ and $J_m(0) = 0$ for $m \geq 1$
5. (Important) $J_m(x) = 0$ has infinitely many solutions. Label these roots, $x = z_{m,i}$, $i = 1, 2, 3, \dots$

7.6.3 Normal Modes of a Circular Membrane

Go back to the problem: circular membrane, radius $r = a$.

Since $y(x) = J_m(x)$ and $R(r) \equiv y(x) = y(r\sqrt{\lambda})$, the general solutions, bounded at $r = 0$ of the $R(r)$ -eqn are given by

$$R(r) = C_m J_m(r\sqrt{\lambda})$$

where C_m an arbitrary constant.

The boundary condition at the edge of the drum gives $R(a) = 0$. So

$$J_m(a\sqrt{\lambda}) = 0. \quad (7.30)$$

Therefore we must have

$$a\sqrt{\lambda} = z_{m,i}, \quad i = 1, 2, \dots$$

where $z_{m,i}$ are the zeros of $J_m(x)$. Hence

$$R(r) = C_{m,i} J_m\left(\frac{z_{m,i} r}{a}\right) \quad \text{for } r \leq a, \quad i = 1, 2, \dots \quad (7.31)$$

with $C_{m,i}$ constants, after modifying the notation.

Hence, the **frequencies** of oscillations are given by

$$\omega/c = \sqrt{\lambda} = z_{m,i}/a, \quad \text{or} \quad \omega_{m,i} = \frac{z_{m,i}c}{a}$$

The modal shape of the membrane comes from reconstructing the solution from its separable parts

$$u(r, \theta, t) = \phi_{m,i}(r, \theta) \cos(\omega_{m,i}t + \delta) = C_{m,i} J_m\left(\frac{z_{m,i} r}{a}\right) [A_m \cos m\theta + B_m \sin m\theta] \cos(\omega_{m,i}t + \delta)$$

where $\omega_{m,i} = z_{m,i}c/a$.

The first few zeros of the Bessel functions (approx)

$$z_{0,1} = 2.4 \dots, \quad z_{1,1} = 3.8 \dots, \quad z_{2,1} = 5.1 \dots, \quad z_{0,2} = 5.5 \dots$$

So the fundamental (lowest-frequency) mode has frequency $\approx 2.4c/a$ where a is the radius of the drum and c is the speed of waves on the drumskin. The larger the radius, the lower the frequency. This is why a bass drum must be big.

7.6.4 The Initial-value problem

In both the rectangular and circular membrane problem, an initial value problem in which u and u_t are specified at $t = 0$, a general solution is formed by the superposition of all possible normal modes. The unknown coefficients can, in principle, be found by applying initial conditions on u and u_t at $t = 0$, but this is too complicated for this course.