Valeriy Slastikov **Autumn, 2005**

4 Signed measures and Radon-Nikodym theorem

Let $(X, \mathfrak{M})$ be some measurable space.

Definition 4.1 *A set function ν is called a signed measure if*

1. $\nu(\varnothing) = 0$;
2. *its domain of definition $\mathfrak{M}$ is a σ-algebra;*
3. *ν is σ-additive.*

Example 4.1

- *If $\nu(A) = \mu_1(A) - \mu_2(A)$, where μ_1 and μ_2 are Lebesgue measures, then ν is a signed measure;*
- *if $\nu(A) = \int_A f(x) d\mu$, where f is integrable function and μ is Lebesgue measure, then ν is a signed measure.*

Definition 4.2 *Let ν be a signed measure on $(X, \mathfrak{M})$. The set $A \in \mathfrak{M}$ is positive with respect to ν if $\nu(A \cap E) \geq 0$ for all $E \in \mathfrak{M}$. The set $B \in \mathfrak{M}$ is negative with respect to ν if $\nu(B \cap E) \leq 0$ for all $E \in \mathfrak{M}$.*

Theorem 4.3 *(Hahn decomposition) Let ν be a signed measure on $(X, \mathfrak{M})$, then there exists a negative set $A^- \in \mathfrak{M}$ and the positive set $A^+ = X \backslash A^-$*

Proof Fact 1. Negative sets form a σ-ring $\mathfrak{N}$. Obviously

- $\varnothing \in \mathfrak{N}$;
- $A \in \mathfrak{N}$, $B \in \mathfrak{N}$ imply $A \cap B \in \mathfrak{N}$;
- $A \in \mathfrak{N}$, $B \in \mathfrak{N}$ imply $A \backslash B \in \mathfrak{N}$.

Since $A \cup B = A \cup (B \backslash A)$, A and $B \backslash A$ are disjoint, we obtain

$$\nu((A \cup B) \cap E) = \nu(A \cap E) + \nu((B \backslash A) \cap B)$$

for any $E \in \mathfrak{M}$. From this formula it is easy to see that if $A, B \in \mathfrak{N}$ then $A \cup B \in \mathfrak{N}$. The same arguments work for a countable union of sets. Therefore $\mathfrak{N}$ is a σ-ring.

Let us define

$$\beta = \inf_{B \in \mathfrak{N}} \nu(B).$$

Obviously $\beta = \lim_{n\to\infty} \nu(B_n)$, $B_n \in \mathfrak{N}$. We define

$$B = \cup_{n=1}^{\infty} B_n.$$

It is possible to show that $\nu(B) = \beta$:

1. $\beta \leq \nu(B)$ since $B \in \mathfrak{N}$;
2. $\nu(B) = \nu(B_n) + \nu(B \setminus B_n) \leq \nu(B_n)$ since $B \setminus B_n \in \mathfrak{N}$. Therefore $\beta = \lim_{n\to\infty} \nu(B_n) \geq \nu(B)$.

We want to show that $A = X \setminus B$ is positive. Suppose not, then there exists $E_0 \subset A$ such that $\nu(E_0) < 0$. If $E_0 \in \mathfrak{N}$ then $B \cup E_0 \in \mathfrak{N}$ and $\nu(B \cup E_0) < \nu(B)$ that contradicts minimality of β. Therefore E_0 is not negative and there exists $C \subset E_0$ such that $\nu(C) > 0$. Now we do the following procedure:

- find a positive number k_1 such that there exists $E_1 \subset E_0$ with $\nu(E_1) \geq k_1$ and k_1 is the maximal of such numbers;
- consider $E_0 \setminus E_1$ ($\nu(E_0 \setminus E_1) = \nu(E_0) - \nu(E_1) < 0$) and find a positive number k_2 such that there exists $E_2 \subset E_0 \setminus E_1$ with $\nu(E_2) \geq k_2$ and k_2 is the maximal of such numbers; it is clear that $k_2 \leq k_1$.

Continue this procedure we find a sequence of disjoint sets $\{E_i\} \subset E_0$ with $\nu(E_i) \geq k_i$ (note that this sequence has to be infinite, otherwise we find larger negative set). Obviously $k_i \to 0$ since otherwise $\nu(\cup_i E_i) = \infty$. We define $F_0 = E_0 \setminus \cup_i E_i$, for any $F \subset F_0$ we must have $\nu(F) \leq 0$ and therefore F_0 is negative, disjoint form B, and $\nu(F_0) = \nu(E_0) - \sum_i \nu(E_i) \leq \nu(E_0) < 0$. This contradicts the minimality of β. Theorem is proved.

This decomposition is not unique. However we may show the following: if there are two Hahn decompositions $X = A_1 \cup B_1$ and $X = A_2 \cup B_2$ (A_1, A_2 are positive and B_1, B_2 are negative) then

$$\nu(A_1 \cap E) = \nu(A_2 \cap E), \quad \nu(B_1 \cap E) = \nu(B_2 \cap E)$$

for any $E \in \mathfrak{M}$. Proof of this fact is left as an exercise.

From this it follows that for any signed measure ν we may define $X = A^+ \cup A^-$ and

$$\nu^+(E) = \nu(A^+ \cap E), \quad \nu^-(E) = -\nu(A^- \cap E).$$

It is easy to see that ν^+, ν^- are σ-additive measures and

$$\nu(E) = \nu^+(E) - \nu^-(E).$$

We proved the following theorem.

Theorem 4.4 *(Jordan decomposition) Any signed measure ν may be represented as a difference of two σ-additive measures ν^+ and ν^-.*

Definition 4.5 $|\nu| = \nu^+ + \nu^-$ *is called the total variation of ν.*

Example 4.2 *Let $\nu(A) = \int_A f(x)d\mu$. Obviously $f(x) = f^+(x) - f^-(x)$ and therefore*

$$\nu(A) = \int_A f^+(x)d\mu - \int_A f^-(x)d\mu = \nu^+(A) - \nu^-(A).$$

Definition 4.6 *Let λ and ν be signed measures on $(X, \mathfrak{M})$ then λ is called absolutely continuous with respect to ν ($\lambda \ll \nu$) if $A \in Mr$ and $|\nu|(A) = 0$ imply $\lambda(A) = 0$.*

Theorem 4.7 *(Radon-Nikodym) Let μ be a σ-additive measure on $(X, \mathfrak{M})$, F be a signed measure on $(X, \mathfrak{M})$ and F is absolutely continuous with respect to μ. Then there exists unique $f \in L^1(X, \mu)$ such that*

$$F(A) = \int_A f(x)d\mu$$

for any $A \in \mathfrak{M}$.

Proof Since any signed measure $F = F^+ - F^-$ and $F \ll \mu$ implies $F^+ \ll \mu$ and $F^- \ll \mu$ (prove it!) it is enough to show theorem for σ-additive measures.

Let us define the following set:

$$K = \{f \text{ is integrable on } X \,:\, f(x) \geq 0, \int_A f(x)d\mu \leq F(A) \text{ for any } A \in \mathfrak{M}\},$$

where $f_n \in K$. We also define

$$M = \sup_{f \in K} \int_X f(x)d\mu = \lim_{n\to\infty} \int_X f_n(x)d\mu.$$

Let $g_n(x) = \max\{f_1(x), ..., f_n(x)\}$, we may show that $g_n \in K$:

1. Obviously $g_n \geq 0$, g_n is integrable.

2. We show that $\int_E g_n(x)d\mu \leq F(E)$ for any $E \in \mathfrak{M}$. For any $E \in \mathfrak{M}$ there exists a collection of disjoint sets $\{E_i\}_{i=1}^n$ such that $E = \cup_{i=1}^n E_i$ and $g_n(x) = f_i(x)$ on E_i (prove it!). Therefore

$$\int_E g_n(x)d\mu = \sum_{i=1}^n \int_{E_i} f_i(x)d\mu \leq \sum_{i=1}^n F(E_i) = F(E).$$

We define $f(x) = \sup_n f_n(x)$, obviously $f(x) = \lim_{n\to\infty} g_n(x)$. By monotone convergence theorem $f \in K$ and $\int_X f(x)d\mu = \lim_{n\to\infty} \int_X g_n(x)d\mu = M$. Define a σ-additive measure

$$\lambda(E) = F(E) - \int_E f(x)d\mu$$

for any $E \in \mathfrak{M}$. We want to show that $\lambda(E) = 0$ for any $E \in \mathfrak{M}$.

Lemma 4.1 *Let ν, μ be σ-additive measures and $\nu \ll \mu$. Then there exists $n \in \mathbb{N}$ and $B \in \mathfrak{M}$ such that $\mu(B) > 0$ and B is positive with respect to $\nu - \frac{1}{n}\mu$.*

Proof Let $X = A_n^- \cup A_n^+$ be Hahn decomposition corresponding to a signed measure $\nu - \frac{1}{n}\mu$. We define $A_0^- = \cap_{n=1}^\infty A_n^-$, $A_0^+ = \cup_{n=1}^\infty A_n^+$ then $A_0^- \cup A_0^+ = X$. We have

$$\nu(A_0^-) \leq \frac{1}{n}\mu(A_0^-) \quad \text{for any } n$$

and therefore $\nu(A_0^-) = 0$. So we obtain $\nu(A_0^+) > 0$ and hence $\mu(A_0^+) > 0$ (since $\nu \ll \mu$). Therefore there exists $n \in \mathbb{N}$ such that $\mu(A_n^+) > 0$ and $\nu(E \cap A_n^+) - \frac{1}{n}\mu(E \cap A_n^+) \geq 0$ for any $E \in \mathfrak{M}$ since A_n^+ is positive with respect to $\nu - \frac{1}{n}\mu$. Lemma is proved.

By definition λ is a σ-additive measure and $\lambda \ll \mu$. Therefore there exists B and $n \in \mathbb{N}$ such that $\lambda(E \cap B) \geq \frac{1}{n}\mu(E \cap B)$ for any $E \in \mathfrak{M}$ and $\mu(B) > 0$. Define

$$h(x) = f(x) + \frac{1}{n}\chi_B(x),$$

then for any $E \in \mathfrak{M}$

$$\begin{aligned}
\int_E h(x)d\mu &= \int_E f(x)d\mu + \frac{1}{n}\mu(E \cap B) \leq \int_E f(x)d\mu + \lambda(E \cap B) \\
&= \int_E f(x)d\mu + F(E \cap B) - \int_{E\cap B} f(x)d\mu \\
&= \int_{E\backslash B} f(x)d\mu + F(E \cap B) \leq F(E\backslash B) + F(E \cap B) = F(E).
\end{aligned}$$

Therefore $h \in K$ and

$$\int_X h(x)d\mu = \int_X f(x)d\mu + \frac{1}{n}\mu(B) > M,$$

so we have a contradiction and therefore for any $E \in \mathfrak{M}$ $F(E) = \int_E f(x)d\mu$. Uniqueness is obvious. Theorem is proved.

5 L^p-spaces

Let f be an integrable function over X. We call $\tilde{f}$ the class of equivalence of functions $g_f(x)$ such that $f(x) = g_f(x)$ a.e. on X. It is easy to show that equality a.e. defines the equivalence relation.

Definition 5.1 *Let $p \geq 1$ and $|f(x)|^p$ is an integrable function over X then $\tilde{f} \in L^p(X, d\mu)$.*

So $L^p(X, d\mu)$ is the space of equivalence classes of all p-integrable functions. Instead of dealing with classes of equivalence we will deal with representatives of these classes, i.e. with usual functions.

Definition 5.2 *The function $\|\cdot\| : V \to \mathbb{R}$ acting on some vector space V is called a norm if for any $f, g \in V$*

1. $\|f\| \geq 0$;
2. $\|f\| = 0$ *if and only if* $f = 0$;
3. $\|\lambda f\| = |\lambda|\|f\|$ *for any* $\lambda \in \mathbb{R}$;
4. $\|f + g\| \leq \|f\| + \|g\|$.

Proposition 5.3 *Let $f \in L^p(X, d\mu)$ then*

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu\right)^{\frac{1}{p}}$$

is a norm of f.

Note that this norm is the same for all functions in the class of equivalence.

Proof We have to check points 1-4. It is easy to see that points 1-3 are satisfied, we show point 4. For $p = 1$ it is obvious. Let us assume $p > 1$.

Lemma 5.1 *If $f \in L^p(X, d\mu)$, $g \in L^q(X, d\mu)$ for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ then*

$$\int_X |fg| d\mu \leq \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} \left(\int_X |g|^q d\mu \right)^{\frac{1}{q}} = \|f\|_p \|g\|_q$$

Proof Proof is left as an exercise.

It is easy to see that if $f, g \in L^p(X, d\mu)$ then $|f + g| \in L^p(X, d\mu)$ (since$|a + b|^p \leq 2^{p-1}(|f|^p + |g|^p)$). Obviously $|f(x) + g(x)|^p \leq |f(x) + g(x)|^{p-1}(|f(x)| + |g(x)|)$ and therefore since $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned}
\int_X |f + g|^p d\mu &\leq \int_X |f + g|^{p-1}|f| d\mu + \int_X |f + g|^{p-1}|g| d\mu \\
&\leq \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} \left(\int_X |f + g|^{(p-1)q} d\mu \right)^{\frac{1}{q}} \\
&+ \left(\int_X |g|^p d\mu \right)^{\frac{1}{p}} \left(\int_X |f + g|^{(p-1)q} d\mu \right)^{\frac{1}{q}}.
\end{aligned}$$

This implies

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proposition is proved.

Definition 5.4 *Complete normed vector space is called Banach space.*

Theorem 5.5 *For $1 \leq p < \infty$ $L^p(X, d\mu)$ is a Banach space.*

Proof Let $\{f_n\}$ be a Cauchy sequence in L^p. We choose a subsequence $\{f_{n_j}\} \subset \{f_n\}$ such that

$$\|f_{n_{j+1}} - f_{n_j}\|_p \leq \frac{1}{2^j}$$

and define

$$G_m(x) = \sum_{j=1}^{m} |f_{n_{j+1}}(x) - f_{n_j}(x)|.$$

Obviously $\|G_m\|_p \leq 1$ and $G_m(x)$ is a monotone increasing sequence. By monotone convergence theorem:

$$G_m(x) \to G_(x) = \sum_{j=1}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| < \infty \quad \text{a.e}$$

and

$$\lim_{m\to\infty}\int_X |G_m(x)|^p d\mu = \int_X |G(x)|^p d\mu \le 1.$$

Therefore $f = \lim_{j\to\infty} f_{n_j}$ exists a.e., $f \in L^p(X, d\mu)$ and

$$\|f - f_{n_k}\|_p = \|\sum_{j=k}(f_{n_{j+1}} - f_{n_j})\|_p \le \sum_{j=k}\|f_{n_{j+1}} - f_{n_j}\|_p \le \frac{1}{2^{(k-1)}}.$$

So we have $f_{n_j} \to f$ in $L^p(X, d\mu)$ and moreover if $n > n_j$

$$\|f_n - f\|_p \le \|f - f_{n_j}\|_p + \|f_{n_j} - f_n\|_p \le 2^{-(j-1)} + 2^{-(j-1)} = 2^{2-j} \to 0$$

as $n > n_j \to \infty$. Theorem is proved.

Theorem 5.6 *(Approximation of L^p) Assume $1 < p < \infty$, then:*

1. *Simple functions are dense in $L^p(X, d\mu)$.*
2. *Elementary functions are dense in $L^p(X, d\mu)$.*
3. *Uniformly continuous functions are dense in $L^p(X, d\mu)$.*

Proof Proof is left as an exercise.

Proposition 5.7 *(Jensen's inequality) Let $f \in L^1(X, d\mu)$ and $\phi : X \to \mathbb{R}$ be convex function. Then*

$$\phi\left(\frac{1}{\mu(X)}\int_X f(x)d\mu\right) \le \frac{1}{\mu(X)}\int_X \phi(f(x))d\mu$$

Proof Proof is left as an exercise.

5.1 Duality

Definition 5.8 *A bounded linear functional on a Banach space B is a mapping $F : B \to \mathbb{R}$ such that*

1. *$F(\alpha f_1 + \beta f_2) = \alpha F(f_1) + \beta F(f_2)$ for any $\alpha, \beta \in \mathbb{R}$ and $f_1, f_2 \in B$;*
2. *$|F(f)| \le C\|f\|$ for any $f \in B$; here C is a constant independent of f and $\|\cdot\|$ is a norm on B.*

Definition 5.9 *A collection of all bounded liner functionals on a Banach space B is called a dual space to B and is denoted by B^*. It is usually endowed with the following norm*

$$\|F\|_* = \sup_{f\in B} \frac{|F(f)|}{\|f\|}. \tag{1}$$

Theorem 5.10 *B^* is a Banach space.*

Proof It is easy to show that B^* is a normed vector space with the norm $\|\cdot\|_*$. We show that B^* is complete. Suppose $\{F_n\}$ is a Cauchy sequence, i.e. $\|F_n - F_m\|_* \to 0$ as $n, m \to \infty$. Since for any $f \in B$

$$|F_n(f) - F_m(f)| \leq \|F_n - F_m\|_* \|f\| \to 0$$

we obtain that $\{F_n(f)\}$ is a Cauchy sequence and hence $F_n(f) \to a \equiv F(f)$. It is easy to check that F is a linear functional (do it!). Using the following inequality

$$|F(f)| = \lim_{n\to\infty} |F_n(f)| \leq \lim_{n\to\infty} \|F_n\|_* \|f\|,$$

and the fact that $\{\|F_n\|\}$ is a Cauchy sequence, we obtain $\|F\| \leq \lim_n \|F_n\| \leq C$. Therefore $|F(f)| \leq C\|f\|$ and F is a bounded linear functional.

Now we want to show that $\|F_n - F\|_* \to 0$ as $n \to \infty$. It is easy to see from

$$\frac{|F_n(f) - F(f)|}{\|f\|} \leq \lim_m \frac{|F_n(f) - F_m(f)|}{\|f\|} \leq \lim_m \|F_n - F_m\|_*.$$

Taking sup over $f \in B$ and then limit as $n \to \infty$ from both sides we obtain the result. Theorem is proved.

Theorem 5.11 *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(L^p(X, d\mu))^* \equiv L^q(X, d\mu)$.*

Proof Step 1. We show that $L^q(X, d\mu) \subset (L^p(X, d\mu))^*$ is an isometric injection. Take any $g \in L^q(X, d\mu)$ and define

$$F_g(f) = \int_X g(x) f(x) d\mu$$

for any $f \in L^p(X, d\mu)$. It is easy to see that F_g is a bounded linear functional on $L^p(X, d\mu)$. Therefore to any $g \in L^q(X, d\mu)$ there corresponds a bounded linear functional $F_g \in (L^p(X,d\mu))^*$. Now we have to show that $\|F_g\|_* = \|g\|_q$. By definition

$$\|F_g\|_* = \sup_{f \in L^p(X,d\mu)} \frac{\int_X f(x)g(x)d\mu}{\|f\|_p}.$$

Using Holder inequality we see that $\|F_g\|_* \leq \|g\|_q$. Taking $f = |g|^{q-1} sgn(g)$ we see that $\|F_g\|_* \geq \|g\|_q$ and therefore $\|F_g\|_* = \|g\|_q$. We showed that $g \to F_g$ is an isometric injection of L^q into $(L^p)^*$.
Step 2. Now we want to show that for any $F \in (L^p(X,d\mu))^*$ there exists $g \in L^q(X,d\mu)$ such that $F(f) = \int_X g(x)f(x)d\mu$ for all $f \in L^p(X,d\mu)$. In the previous step we proved that for any $g \in L^q(X,d\mu)$

$$\|g\|_q = \|F_g\|_* = \sup_{f \in L^p(X,d\mu)} \frac{\int_X f(x)g(x)d\mu}{\|f\|_p}.$$

It is easy to show that

$$\|g\|_q = \sup\left\{\int_X f(x)g(x)d\mu, f \in L^p(X,d\mu),\ \|f\|_p \leq 1 \text{ and } f \text{ is simple}\right\}$$

(prove it!) Take any$F \in (L^p(X,d\mu))^*$ and define a set function

$$\nu(A) = F(\chi_A)$$

for any $A \in \mathfrak{M}$. Let's check that ν is a signed measure:

1. $\nu(\varnothing) = F(0) = 0$;
2. if $A \cap B = \varnothing$ then $\nu(A \cup B) = F(\chi_{A\cup B}) = F(\chi_A + \chi_B) = F(\chi_A) + F(\chi_B) = \nu(A) + \nu(B)$;
3. if $A_n \uparrow A$ then $\chi_{A_n} \to \chi_A$ in $L^p(X,d\mu)$ and hence $|F(\chi_{A_n} - \chi_A)| \leq C\|\chi_{A_n} - \chi_A\|_p \to 0$. This obviously implies $\nu(A_n) \to \nu(A)$ and this implies countable additivity of ν (prove it!).

Therefore ν is a signed measure. If $A \in \mathfrak{M}$ and $\mu(A) = 0$ then $\chi_A = 0$ a.e. and therefore $\nu(A) = F(\chi_A) = 0$ and we obtain $\nu \ll \mu$. Using Radon-Nikodym theorem we have for any $A \in \mathfrak{M}$

$$\nu(A) = \int_A g(x)d\mu,$$

where g is some integrable function. Our goal is to show that $g \in L^q(X, d\mu)$. Let $s(x) = \sum_{i=1}^n a_i \chi_{A_i}(x)$ be a simple function. Then

$$F(s) = F(\sum_{i=1}^n a_i \chi_{A_i}) = \sum_{i=1}^n a_i F(\chi_{A_i}) = \sum_{i=1}^n a_i \nu(A_i)$$
$$= \sum_{i=1}^n a_i \int_{A_i} g(x) d\mu = \int_X g(x)s(x) d\mu$$

It is easy to see that

$$\sup \left\{ \int_X s(x)g(x)d\mu, s \in L^p(X, d\mu),\ \|s\|_p \le 1 \text{ and } s \text{ is simple } \right\} =$$
$$= \sup \{F(s), s \in L^p(X, d\mu),\ \|s\|_p \le 1 \text{ and } s \text{ is simple } \} \le \|F\|_*.$$

Therefore $g \in L^q(X, d\mu)$ and $\|g\|_q \le \|F\|_*$. Now take any $f \in L^p(X, d\mu)$, we can approximate it by a sequence of simple functions: $s_n \to f$ in $L^p(X, d\mu)$. We know that

$$\int_X g(x)s_n(x)d\mu = F(s_n),$$

$F(s_n) \to F(f)$ and $\int_X g(x)s_n(x)d\mu \to \int_X g(x)f(x)d\mu$ and therefore

$$\int_X g(x)f(x)d\mu = F(f)$$

for any $f \in L^p(X, d\mu)$. Theorem is proved.

5.2 Hilbert space $L^2(X, d\mu)$

Definition 5.12 *Let H be a normed vector space. We call a function $(\cdot, \cdot) : H \times H \to \mathbb{R}$ an inner product if*

1. $(f, g) = (g, f)$ *for any* $f, g \in H$;
2. $(f_1 + f_2, g) = (f_1, g) + (f_2, g)$ *for any* $f_1, f_2, g \in H$;
3. $(\lambda f, g) = \lambda(f, g)$ *for any* $\lambda \in \mathbb{R},\ f, g \in H$;
4. $(f, f) > 0$ *if* $f \neq 0$..

Definition 5.13 *A Banach space H with an inner product $(\cdot, \cdot)$ and a norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$ is called a Hilbert space.*

It is not difficult to show that $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ is actually a norm (do it!)

Exercise 5.1 *Check that R^n and $L^2(X, d\mu)$ are Hilbert spaces.*

Proposition 5.14 *Let H be a Hilbert space. Then for any $f, g \in H$*

1. $|(f,g)| \leq \|f\|\|g\|$;
2. $\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$.

Proof Proof is left as an exercise.

Proposition 5.15 *Let H be a Banach space. If for any $f, g \in H$ $\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$ then H is a Hilbert space.*

Proof Proof is left as an exercise.

Definition 5.16 *Let H be a Hilbert space, for $f, g \in H$ we say that f is orthogonal to g if $(f,g) = 0$. A set $A \subset H$ is called orthogonal set if for any $f, g \in A$ $(f,g) = 0$. A set $A \subset H$ is called orthonormal set if for any $f, g \in A$ $(f,g) = 0$ and $\|f\| = \|g\| = 1$.*

Definition 5.17 *A set $A = \{f_1, ..., f_n, ...\} \subset H$ is linearly independent if $\sum_{i=1}^n \alpha_i f_i = 0$ implies $\alpha_1 = ... = \alpha_n = 0$ for any finite subset of A.*

Proposition 5.18 *An orthonormal set is always linearly independent.*

Proof Proof is left as an exercise.

Definition 5.19 *An orthonormal set $A \subset H$ is called complete if $(f, \phi) = 0$ for all $\phi \in A$ and fixed $f \in H$ implies $f = 0$.*

Definition 5.20 *A Banach space H is called separable if there exists a countable dense subset $E \subset H$.*

Proposition 5.21 *Let $A = \{\phi_1, \phi_2, ...\}$ be an orthonormal set in a separable Hilbert space H. Then A is at most countable.*

Proof For any $\phi, \psi \in A$ we have $\|\phi - \psi\| = \sqrt{2}$. Since H is separable there exists dense and countable subset $E \subset H$. Therefore there exists $f \in E$ and $g \in E$ such that $\|f - \phi\| < \frac{1}{\sqrt{2}}$ and $\|g - \psi\| < \frac{1}{\sqrt{2}}$. By triangle inequality

$$\|\phi - \psi\| \leq \|f - g\| + \|f - \phi\| + \|g - \psi\|$$

and therefore $\|f - g\| > 0$. So we have if $\phi \neq \psi$ then $f \neq g$ and hence if A is uncountable then E is uncountable, but E is at most countable therefore A is at most countable. Proposition is proved.

Theorem 5.22 *$L^2(X, d\mu)$ is a separable Hilbert space.*

Proof Proof is left as an exercise.

Theorem 5.23 *(Riesz - Fisher) Let $\{\phi_n\}$ be an arbitrary orthonormal set in $L^2(X, d\mu)$ and let the corresponding set $\{c_n\} \subset \mathbb{R}$ satisfy $\sum_n c_n^2 < \infty$. Then there exists $f \in L^2(X, d\mu)$ such that*

1. *$c_n = (f, \phi_n)$ for all n;*
2. *$f = \sum_n c_n \phi_n$;*
3. *$\|f\|^2 = \sum_n c_n^2$.*

Proof If $\{\phi_n\}$ is a finite set the result is obvious. Since $\{\phi_n\}$ is at most countable we assume it is infinite. We set $f_n = \sum_{k=1}^n c_k \phi_k$. Obviously we have $\|f_{n+m} - f_n\|^2 = \sum_{k=n+1}^{n+m} c_k^2$. Since $\sum_k c_k^2 < \infty$ the sequence $\{f_n\}$ is a Cauchy sequence. Using completness of L^2 we obtain $f_n \to f$ in $L^2(X, d\mu)$. We claim that this f satisfies 1-3. By construction $f = \sum_{k=1}^\infty c_k \phi_k$. For a fixed ϕ_i we have $(f, \phi_i) = (f_n, \phi_i) + (f - f_n, \phi_i)$. If $n \geq i$ then $(f, \phi_i) = c_i + (f - f_n, \phi_i)$. Since $(f - f_n, \phi_i) \leq \|f_n - f\| \to 0$ as $n \to \infty$ we obtain $(f, \phi_i) = c_i$. Now $\|f - f_n\|^2 = \|f\|^2 - \sum_{k=1}^n c_k^2$ and taking a limit as $n \to \infty$ we obtain $\|f\| = \sum_{k=1}^\infty c_k^2$. Theorem is proved.

Theorem 5.24 *Let $\{\phi_n\}$ be a complete orthonormal set in $L^2(X, d\mu)$ then any $f \in L^2(X, d\mu)$ admits an expansion*

$$f = \sum_{n=1}^{\infty} (f, \phi_n) \phi_n$$

Proof Proof is left as an exercise.