

1 Harmonic Functions

In this section we investigate a very special class of functions called *harmonic*. We will be concentrating on harmonic functions in $\mathbb{R}^2$ but the results of this section are valid in $\mathbb{R}^n$ and most proofs are transferrable directly to $\mathbb{R}^n$.

DEFINITION 1.1 Let $n \in \mathbb{N}^+$ and $\Omega \subset \mathbb{R}^n$ be an open connected set. A function $u \in C^2(\Omega)$ is called **harmonic** if $\Delta u(x) = 0$ for all $x \in \Omega$.

The basic examples of harmonic functions are

EXAMPLE 1.2 Here are several examples of harmonic functions in $\mathbb{R}^2$:

1. $u(x, y) = x + y$, $\Omega \subset \mathbb{R}^2$ is any open set;
2. $u(x, y) = x^2 - y^2$, $\Omega \subset \mathbb{R}^2$ is any open set;
3. $u(x, y) = \ln(x^2 + y^2)$, $\Omega \subset \mathbb{R}^2$ is any open set that does not include 0.

Note that in the third example if Ω contains 0 then $u(x, y)$ will not be harmonic as it is not in $C^2(\Omega)$.

1.1 Mean Value Property

Harmonic functions have many very nice properties. Here we prove that harmonic functions satisfy *the mean value property (MVP)*. We always denote by $B_r(x)$ a ball of radius r centered at a point x .

DEFINITION 1.3 Let $\Omega \subset \mathbb{R}^2$ be an open connected set and $u \in C(\Omega)$. We say that

1. u satisfies the **first mean value property** in Ω if

$$u(x) = \frac{1}{2\pi r} \int_{\partial B_r(x)} u(y) dS_y, \quad \text{for any } B_r(x) \subset \Omega.$$

Note that we can rewrite the above as

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta, \quad (1)$$

where $\hat{n}(\theta) = (\cos \theta, \sin \theta)$, $r > 0$ is such that $B_r(x) \subset \Omega$.

2. u satisfies the **second mean value property** in Ω if

$$u(x) = \frac{1}{\pi r^2} \int_{B_r(x)} u(y) dy, \quad \text{for any } B_r(x) \subset \Omega.$$

Note that we can rewrite the above as

$$u(x) = \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r u(x + s\hat{n}(\theta)) s ds d\theta, \quad (2)$$

where $\hat{n}(\theta) = (\cos \theta, \sin \theta)$, $r > 0$ is such that $B_r(x) \subset \Omega$.

Now we show that the first and the second MVPs are equivalent and therefore later we refer to both of them as mean value property.

PROPOSITION 1.4 *A function $u \in C(\Omega)$ satisfies the first mean value property if and only if it satisfies the second mean value property.*

Proof. Assume u satisfies the first mean value property. Take any $r > 0$ such that $B_r(x) \subset \Omega$, then for all $0 < s < r$ we have

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + s\hat{n}(\theta)) d\theta.$$

Multiplying both parts by $2\pi s$ and integrating from 0 to r we obtain

$$\pi r^2 u(x) = \int_0^{2\pi} \int_0^r u(x + s\hat{n}(\theta)) s ds d\theta.$$

Hence we obtain the second mean value property.

Assume now that the second mean value property is satisfied. Take any $r > 0$ such that $B_r(x) \subset \Omega$, then we have

$$\pi r^2 u(x) = \int_0^{2\pi} \int_0^r u(x + s\hat{n}(\theta)) s ds d\theta.$$

Differentiating with respect to r we obtain

$$2\pi r u(x) = \int_0^{2\pi} u(x + r\hat{n}(\theta)) r d\theta.$$

Dividing both parts by $2\pi r$ we obtain the result. $\square$

Now we would like to show that functions satisfying mean value property and harmonic functions are indeed the same. We first proof the following result.

PROPOSITION 1.5 *Let $\Omega \subset \mathbb{R}^2$ be open connected domain and $u \in C^2(\Omega)$ be a harmonic function. Then u satisfies mean value property in Ω .*

Proof. Let $x \in \Omega$ be any point. We take any $r > 0$ such that $B_r(x) \subset \Omega$ and want to show that if $\Delta u = 0$ in Ω then

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta.$$

In order to prove this result we will use integration by parts or Green's identity. The divergence (or Gauss) theorem tells us that for any vector field $F(x)$ defined for $x \in \Omega$ and any $\Omega_0 \subset \Omega$ we have

$$\int_{\Omega_0} \operatorname{div} F(x) dx = \int_{\partial\Omega_0} F(x) \cdot \hat{n} dS(x),$$

where n is a normal vector to Ω_0 . Now we take $\Omega_0 = B_r(0) \setminus B_\epsilon(0)$. Since $\Delta u = 0$ and $\Delta \ln |x| = 0$ for $x \neq 0$ we have the following equality

$$\begin{aligned} 0 &= \int_{\Omega_0} u(x+y) \Delta \ln |y| - \ln |y| \Delta u(x+y) dy = \int_{\Omega_0} \operatorname{div}(u(x+y) \nabla \ln |y| - \ln |y| \nabla u(x+y)) dy \\ &= \int_{\partial B_r(0)} u(x+y) \nabla \ln |y| \cdot \hat{n} dS(y) - \int_{\partial B_\epsilon(0)} u(x+y) \nabla \ln |y| \cdot \hat{n} dS(y) + \\ &\quad \int_{\partial B_r(0)} \ln |y| \nabla u(x+y) dS(y) - \int_{\partial B_\epsilon(0)} \ln |y| \nabla u(x+y) dS(y) \quad (3) \end{aligned}$$

We can simplify the right hand side to obtain

$$\int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta - \int_0^{2\pi} u(x + \epsilon\hat{n}(\theta)) d\theta = 0.$$

Here we used the fact that

$$\int_{\partial B_r(0)} \ln |y| \nabla u(x + y) dS(y) = \ln |y| \int_{\partial B_r(0)} \nabla u(x + y) dS(y) = -\ln |y| \int_{\partial B_r(0)} \Delta u(x + y) dy = 0.$$

Now we have

$$\int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta = \int_0^{2\pi} u(x + \epsilon\hat{n}(\theta)) d\theta$$

and taking $\epsilon \rightarrow 0$ we obtain

$$\int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta = 2\pi u(x).$$

So we prove the result. $\square$

We can also give a different proof of this result.

Proof 2. Let $x \in \Omega$ be any point. We take any $r > 0$ such that $B_r(x) \subset \Omega$ and want to show that if $\Delta u = 0$ in Ω then

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta.$$

We are using integration by parts

$$\begin{aligned} 0 &= \int_{B_r(x)} \Delta u(y) dy = \int_{B_r(0)} \Delta u(x + y) dy = \int_{\partial B_r(0)} \nabla u(x + y) \cdot \hat{n} dS(y) \\ &= r \int_0^{2\pi} \frac{\partial}{\partial r} u(x + r\hat{n}(\theta)) d\theta = r \frac{\partial}{\partial r} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta. \end{aligned} \quad (4)$$

Therefore we have

$$\int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta = \int_0^{2\pi} u(x) d\theta = 2\pi u(x).$$

The result follows. $\square$

Now we understand that harmonic functions satisfy mean value property and want to prove the opposite result.

PROPOSITION 1.6 *Let $\Omega \subset \mathbb{R}^2$ be open connected domain and $u \in C^2(\Omega)$ satisfies mean value property in Ω . Then u is a harmonic function.*

Proof. Let $x \in \Omega$ be any point. Want to show that if

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta$$

for all $B_r(x) \subset \Omega$ then $\Delta u(x) = 0$. Using formula (5) we see that for all $B_r(x)$

$$\begin{aligned} \int_{B_r(x)} \Delta u(y) dy &= \int_{B_r(0)} \Delta u(x + y) dy = \int_{\partial B_r(0)} \nabla u(x + y) \cdot \hat{n} dS(y) \\ &= r \int_0^{2\pi} \frac{\partial}{\partial r} u(x + r\hat{n}(\theta)) d\theta = r \frac{\partial}{\partial r} \int_0^{2\pi} u(x + r\hat{n}(\theta)) d\theta = 2\pi r \frac{\partial}{\partial r} u(x) = 0. \end{aligned} \quad (5)$$

Now we take an average and let $r \rightarrow 0$ to obtain

$$0 = \lim_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{B_r(x)} \Delta u(y) dy = \Delta u(x).$$

The result is proved. $\square$

1.2 Maximum Principle

In this section we prove a maximum principle for harmonic functions. We start with the following result.

PROPOSITION 1.7 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a harmonic function defined on Ω . Assume u achieves maximum at a point $x_0 \in \Omega$ then $u(x) \equiv \text{const}$ for all $x \in \Omega$.*

Proof. Since $x_0 \in \Omega$ and Ω is an open set we can find $r > 0$ such that $B_r(x_0) \subset \Omega$ and therefore by mean value property we have

$$u(x_0) = \frac{1}{\pi r^2} \int_{B_r(x_0)} u(x) dx.$$

Since $u(x_0) \geq u(x)$ for all $x \in \Omega$ the only way to satisfy mean value property is to have $u(x) = u(x_0)$ for all $x \in B_r(x_0)$. Now take any point $x_n \in \Omega$, we want to show that $u(x_n) = u(x_0)$. We can connect x_0 and x_n by a continuous curve that we cover by intersecting balls $B_{r_0}(x_i) \subset \Omega$ with $2r_0 < r$ in such a way that $|x_{i+1} - x_i| < r_0$ for $i = 0, \dots, n-1$. By the first step we already know that $u(x_1) = u(x_0)$, repeating the arguments we obtain that

$$u(x_0) = u(x_i), \quad \text{for all } i = 1, \dots, n.$$

Therefore we have the result. $\square$

Changing u to $-u$ we obtain the following result.

COROLLARY 1.8 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a harmonic function defined on Ω . Assume u achieves minimum at a point $x_0 \in \Omega$ then $u(x) \equiv \text{const}$ for all $x \in \Omega$.*

Note that we only used mean value property to prove maximum principle. Using above results we clearly have

PROPOSITION 1.9 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a harmonic function defined on Ω . Then u attains its minimum and maximum values over $\bar{\Omega}$ on $\partial\Omega$.*

Now we want to use maximum principle to show some uniqueness properties of harmonic functions.

PROPOSITION 1.10 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a harmonic function defined on $\bar{\Omega}$. Then u is uniquely defined by its values on the boundary $\partial\Omega$.*

Proof. Assume there is a harmonic function v defined on $\bar{\Omega}$ and such that $v(x) = u(x)$ for all $x \in \partial\Omega$. It is clear that a difference $w(x) = u(x) - v(x)$ is also a harmonic function. Moreover, $w(x) = 0$ for all $x \in \partial\Omega$. Since we know that minimum and maximum of harmonic function are achieved at the boundary by obtain $0 \leq w(x) \leq 0$ for all $x \in \bar{\Omega}$. Therefore $w(x) \equiv 0$ and $v(x)$ coincides with $u(x)$. $\square$

COROLLARY 1.11 *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and u be a solution of a Poisson equation*

$$\Delta u(x) = f(x) \quad \text{for all } x \in \Omega, \quad u(x) = u_0(x) \quad \text{for all } x \in \partial\Omega.$$

Then u is unique solution.

Proof. Assume there are two solutions u and v . Then their difference $w(x) = u(x) - v(x)$ is harmonic and zero on $\partial\Omega$. Therefore $w(x) \equiv 0$ and we have a contradiction. $\square$