Lecture 1

Motivation

Riemann Integral

Def Let $f : [a,b] \longrightarrow \mathbb{R}$, $a<b$ and let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$ with $a = x_0 < x_1, \dots < x_n = b$.

We define

$$L(P,f) = \sum_{i=1}^{n} \inf\{f(x) : x_{i-1} \le x < x_i\}(x_i - x_{i-1})$$

and

$$U(P,f) = \sum_{i=1}^{n} \sup\{f(x) : x_{i-1} \le x < x_i\}(x_i - x_{i-1})$$

We say that f is Riemann integrable if and only if

$$\sup_P L(P,f) = \inf_P U(P,f),$$

where we take sup & inf over all possible Riemann integral partitions. We denote as

$$\int_a^b f(x)dx = \sup_P L(P,f) = \inf_P \{U(P,f)\}.$$

Theorem A bounded function is Riemann integrable if and only if $\forall \varepsilon > 0$ $\exists$ partition P on $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon$$

Proof Assume that condition holds, i.e. for $\forall \varepsilon > 0$ $\exists P$: $U(f,P) - L(f,P) < \varepsilon$.

It is clear that

$$\inf_P U(P,f) \le U(f,P) \Leftarrow \text{for any } P$$

also $\sup_P L(P,f) \ge L(f,P)$

On the other hand we have

$$\inf_P U(P,f) \geq \sup_P L(P,f)$$

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proof Take two partitions P_1 & P_2

then $L(P_1,f) \leq U(P_2,f)$ since we can alway take a refinement $\{P_1,P_2\}=Q$ containing all points of P_1 & P_2 and

$L(Q,f) \geq L(P_1,f)$ & $U(Q,f) \leq U(P_2,f)$

So $L(P_1,f) \leq L(Q,f) \leq U(Q,f) \leq U(P_2,f)$.

Now we have

$$L(P_1,f) \leq U(P_2,f) \quad \forall P_1, P_2$$

take sup on the left & inf on the right to obtain

$$\sup_P L(P,f) \leq \inf_P U(P,f).$$

$$0 \leq \inf_P U(P,f) - \sup_P L(P,f) \leq U(f,P) - L(f,P) \leq \varepsilon$$

Since ε is arbitrary we obtain

$$0 \leq \inf_P U(P,f) - \sup L(P,f) \leq 0$$

Now assume f is Riemann integrable $\Rightarrow$

$$\Rightarrow \inf_P U(P,f) = \sup_P \{L(P,f)\}$$

Pick any $\varepsilon>0$ then $\exists P_1$ such that

$$U(P_1,f) \leq \inf_P U(P,f) + \frac{\varepsilon}{2}$$

and $\exists P_2$: $L(P_2, f) \geq \sup_P L(f,P) - \frac{\varepsilon}{2}$

Let $Q = \{P_1, P_2\}$ be a refinement of $P_1 \& P_2$

then

$$0 \leq U(f,Q) - L(f,Q) \leq$$

$$\leq U(f,P_1) - L(f,P_2) \leq \inf_P U(f,P) -$$

$$- \sup_P L(f,P) + \varepsilon = \varepsilon$$

$\blacksquare$

Theorem above gives a good idea on how one can check integrability of a function.

$\boxed{\text{What kind of functions can we integrate using Riemann integral?}}$

__Theorem__ A bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuities of f has __measure zero__.

__Def__ The Lebesgue measure of an open interval $I = (a,b)$ is $\mu(I) = b-a$.

A set $E \subset \mathbb{R}$ has measure zero if $\forall \varepsilon > 0$ $\exists$ a countable collection of intervals $\{I_1, I_2, \ldots\}$ such that

$$E \subset \bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \sum_{i=1}^{\infty} \mu(I_i) < \varepsilon.$$

__Basic facts__ :
* A subset of a set of measure zero has measure zero.
* A countable union of sets of measure zero has measure zero.

Lecture 2 Motivation

We have a result characterising Riemann integrable functions. How bad is it?

Example If you change a constant function on a countable set the function might not be Riemann integrable anymore.

Take $f(x) = \begin{cases} 1 & x \in [a,b] \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \cap [a,b] \end{cases}$

It is constant except for a countable set Q. The Riemann integral of f does not exist. Check it!

Example Take a sequence of Riemann integrable functions $\{f_n(x)\}$ and let

$f_n(x) \longrightarrow f(x)$ pointwise

It's not difficult to find a sequence $\{f_n\}$ so that it's limit is not Riemann integrable.

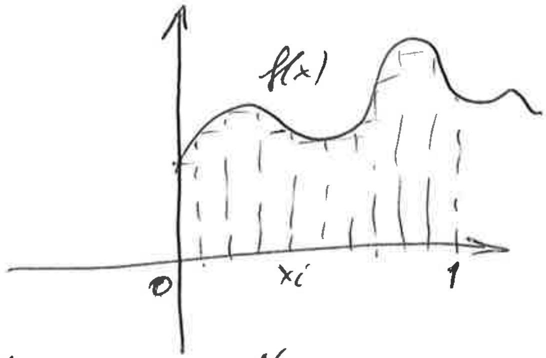
Do it!

Example If $|f(x)|$ is Riemann integrable does not mean $f(x)$ is Riemann integrable.

So the basic deficiencies of Riemann integral are

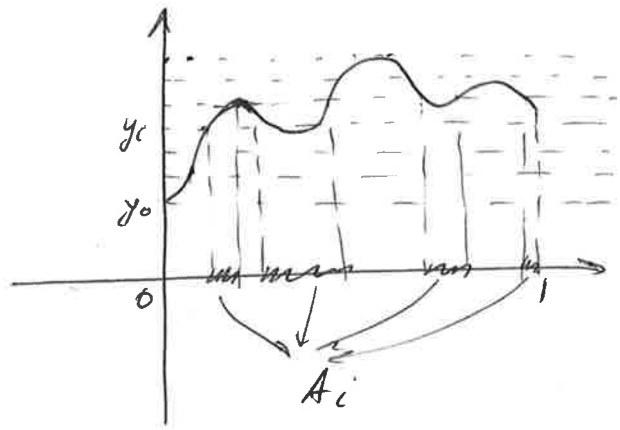
1) Change a function on a "very small set" and it becomes not integrable
2) Pointwise convergence does not preserve integrability. One needs uniform convergence for this (well, almost)
3) Fundamental theorem of calculus?

The problems of RI come from the fact that we split the base space $[a,b]$ to construct an integral sum. What if we split the target space (image of f)?



$$\int_0^1 f(x)\,dx \sim \sum_{i=1}^{N} f(\xi_i)(x_i - x_{i-1})$$

$$\xi_i \in (x_{i-1}, x_i)$$



$$A_i = \{x \in [0,1) : y_i \le f(x) < y_{i+1}\}$$

$$\int_0^1 f(x)\,dx \sim \sum_{i=1}^{N} y_i^* \text{ "length}(A_i)\text{"}$$

$$y_i \le y_i^* < y_{i+1}$$

Why it was difficult to RI some functions? Well, the image might change drastically on a very small set. It seems that new notion controls it as you control the splitting of the image of f. However the sets A_i that you obtain might be very weird. So you have to find a way to measure these sets.

Here comes your measure theory.

Some strange sets:

a) Cantor set — measurable
b) Vitali's set — non-measurable

3 Cantor set

Construction: ① Take $[0,1]$, divide into 3 equal parts and remove a middle one $\Rightarrow F_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

(you remove an open interval).

② Repeat the process for each of the closed ~~subin~~ subintervals of F_1 to obtain:

$$F_2 = [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1]$$

③ Repeat the process for each of the subintervals of F_2, etc.

We see that at each step k we delete 2^{k-1} and are left with a union of 2^k closed sets intervals of length $(\frac{1}{3})^k$

Cantor set is defined as $C = \bigcap_{k\geq 1}^{\infty} F_k$.

Properties of Cantor set:

1) C is closed
2) The length of removed intervals is
$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = 1$$
3) Cantor set has measure zero.

It is clear that $C \subset F_N = \bigcup_{i=1}^{2^N} J_i \quad \forall N \geq 1$ (J_i are closed of length $\frac{1}{3^N}$)

Pick any $\varepsilon > 0$ and fix ~~[illegible]~~ $1 > \delta > 0$ ~~[illegible]~~

For any N: $F_N \subset \bigcup_{i=1}^{2^N} I_i$ with I_i being open and containing J_i (just pick the length of I_i to be $\left(\frac{1}{3-\delta}\right)^N$) $\sum_{i=1}^{2^N} \mu(I_i) = \left(\frac{2}{3-\delta}\right)^N$

Now pick N large enough so that $\left(\frac{2}{3-\delta}\right)^N < \varepsilon$

4) It is clear that Cantor set contains all end points of F_N $\forall N \geq 1$

It is also clear that Cantor set does not contain any open interval (if it were ~~...~~ it would not have measure zero)

5) Cardinality of Cantor set is the same as cardinality of $[0,1]$, i.e. $\exists$ 1-1 correspondence between Cantor set & $[0,1]$.

Sketch:

Let us try to "number" the intervals in F_N

$$F_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1] = S_0 \cup S_2$$

We use "0" for left interval and "2" for right interval

$$F_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1] =$$
$$= S_{00} \cup S_{02} \cup S_{20} \cup S_{22}$$

If we continue, we obtain that for any infinite string of numbers containing only 0 & 2

$a_1 a_2 a_3 \ldots$

$$S_{a_1} \supset S_{a_1 a_2} \supset S_{a_1 a_2 a_3} \supset \ldots$$

Taking intersection of these intervals we obtain a point in a Cantor set (by definition of a set).

Moreover, different points in C cannot correspond to the same string $a_1 a_2 a_3 \ldots$ and two different strings cannot give the same point. $\Rightarrow$ 1-1 correspondence. Change $2 \to 1 \Rightarrow [0,1]$ ∎

Measurable sets and functions

We now take a formal (but a bit vague) approach to define sets that we can "measure" and functions on these sets.

Def Let $X \neq \emptyset$ be a set and $\underline{X}$ be a family of subsets of X.
We say the $\underline{X}$ is a σ-algebra if

1) $\emptyset \in \underline{X}$, $X \in \underline{X}$
2) $A \in \underline{X} \Rightarrow A^c \in \underline{X}$
3) $A_1, A_2, \ldots, A_n, \ldots \in \underline{X} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \underline{X}$

Note that from definition it is clear that $A, B \in \underline{X} \Rightarrow A \cap B \in \underline{X}$ and the same hold for a countable family $\{A_i\}_{i=1}^{\infty}$.
We also call a pair $(X, \underline{X})$ a <u>measurable space</u>, all sets in $\underline{X}$ are called <u>measurable</u>

Def Let $\cancel{X} \neq \emptyset$ and $\mathcal{A}$ be a collection of subsets of $\cancel{X}$, Let $\underline{Y}$ be a collection of σ-algebras containing $\mathcal{A}$. Then

$$\beta(\mathcal{A}) = \bigcap_{\underline{X} \in \underline{Y}} \underline{X}$$

is the σ-algebra generated by $\mathcal{A}$. This is the smallest σ-algebra containing $\mathcal{A}$.

Defn Let $X = \mathbb{R}$ and $\mathcal{A} = \{(a,b),\ a, b \in \mathbb{R},\ a < b\}$
The σ-algebra generated by $\mathcal{A}$ is called the <u>Borel algebra</u> $\mathcal{B}$. If $B \in \mathcal{B}$ then B is called a <u>Borel set</u>.

Let $X = \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ If E is Borel set and
$E_1 = E \cup \{-\infty\}$, $E_2 = E \cup \{+\infty\}$, $E_3 = E \cup \{\pm\infty\}$ $\forall E \in \mathcal{B}$
let $\overline{\mathcal{B}}$ be collection of all sets E, E_1, E_2, E_3
then $\overline{\mathcal{B}}$ is extended Borel algebra.

Definition Let $(X, \underline{X})$ be a measurable space. Then $f : X \to \mathbb{R}$ is an $\underline{X}$-measurable function if $f^{-1}(A) \in \underline{X}$ for any Borel set $A \in \mathcal{B}$.

When we are talking about functions with image in $\mathbb{R}$ we usually are considering Borel algebra on $\mathbb{R}$.

If we have a function $f : X \to Y$ and spaces $(X, \underline{X})$, $(Y, \underline{Y})$ are measurable then f is measurable if $f^{-1}(A) \in \underline{X}$ for all $A \in \underline{Y}$.

Lemma A function $f : X \to \mathbb{R}$ is measurable if and only if for any $c \in \mathbb{R}$ the set
$$\{x \in X : f(x) < c\}$$
is measurable.

Proof Necessity is trivial.
To prove sufficiency we have to show the σ-algebra generated by sets $(-\infty, c)$ is actually Borel algebra on $\mathbb{R}$. Let $\underline{Y}$ is gen. by
We take 2 sets $(-\infty, a)$ & $(-\infty, b)$ with $b > a$
~~_______~~ then it's clear $[a, b) \in \underline{Y}$
$\forall$ any $a, b \in \mathbb{R}$ $a < b$ $\Rightarrow$ $\bigcup_{n=1}^{\infty} [a - \frac{1}{n}, b) = (a, b) \in \mathbb{R}$
$\Rightarrow$ $\underline{Y}$ contains all sets of form (a, b) $\Rightarrow$
$\Rightarrow$ it's Borel σ-algebra.

If $A = \{x \in X : f(x) < c\}$ is measurable then $f^{-1}(A) \in \mathcal{B}$ and hence varying c we can construct a minimal σ-algebra $\mathcal{U}$ containing all sets $f^{-1}(\{x \in X : f(x) < c\})$.

It follows that $\mathcal{Z} \subset \mathfrak{X}$ or

$$\beta(f^{-1}(-\infty,c)) \subset \mathfrak{X} \quad \text{But}$$

$$\beta(f^{-1}(-\infty,c)) = f^{-1}(\beta((-\infty,c)))$$ show it.

so we are done.

Lemma Let $f: X \to \mathbb{R}$ be some function. Then the following statements are equivalent

1. $\{x \in X : f(x) < c\} \in \mathfrak{X} \quad \forall c \in \mathbb{R}$
2. $\{x \in X : f(x) \leq c\} \in \mathfrak{X} \quad \forall c \in \mathbb{R}$
3. $\{x \in X : f(x) > c\} \in \mathfrak{X} \quad \forall c \in \mathbb{R}$
4. $\{x \in X : f(x) \geq c\} \in \mathfrak{X} \quad \forall c \in \mathbb{R}$

Proof Excercise.

Def Let $f: X \to \overline{\mathbb{R}}$ (extended)
Then f is measurable iff
$$\{x \in X : f(x) > \alpha\} \in \mathfrak{X} \quad \forall \alpha \in \mathbb{R}$$ (not extended)

It's enough to consider $\alpha \in \mathbb{R}$ since we can always recover sets
$$\{x \in X : f(x) = +\infty\}$$
& $\{x \in X : f(x) = -\infty\}$
by taking countable intersections.

(I)

Lecture 4

Measurable functions

Lemma Let $f: X \to \overline{\mathbb{R}}$ be $\mathfrak{X}$-measurable and $\phi: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be $\overline{\mathcal{B}}$-measurable.
Then $\phi(f(x))$ is $\mathfrak{X}$-measurable.

Proof $g(x) = \phi(f(x))$. Take any $A \in \overline{\mathcal{B}}$
then $g^{-1}(A) = f^{-1}(\phi^{-1}(A))$
Since $\phi^{-1}(A) \in \overline{\mathcal{B}}$ we have $f^{-1}(\phi^{-1}(A)) \in \mathfrak{X}$,
so $g^{-1}(A) \in \mathfrak{X}$.

Lemma Let $f: X \to \overline{\mathbb{R}}$ & $g: X \to \overline{\mathbb{R}}$ be measurable functions then
1) $\alpha f + \beta g$ is measurable
2) $f \cdot g$ & $\frac{f}{g}$ $(g \neq 0)$ are measurable
3) $\max(f,g)$ & $\min(f,g)$ are measurable
4) If $F: \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}$ is continuous then

Proof If we prove 4) we are done.
Fix any $\alpha \in \mathbb{R}$ we have to show that
$$X_\alpha = \{x : F(f(x), g(x)) < \alpha\} \text{ is measurable}$$
It is clear that
$$X_\alpha = \{x : (f(x), g(x)) \in F^{-1}((-\infty, \alpha))\}$$
We define $A = F^{-1}((-\infty, \alpha))$, since F is continuous A is open.
Any open set can be represented as a countable union of open rectangles $(a,b) \times (c,d)$.
Hence it is enough to show that
$$\{x : (f(x), g(x)) \in (a,b) \times (c,d)\} \text{ is}$$
measurable $= f^{-1}(a,b) \cap g^{-1}(c,d)$

② Note that a set of measurable functions $f: X \to \bar{\mathbb{R}}$ is a vector space. We denote it as $M(X, \mathfrak{X})$.
($M(X,\mathfrak{X})$ is extended space)

Lemma Let $\{f_n\}_{n\geq 1}^{\infty}$ be a sequence of $\mathfrak{X}$-measurable functions $(\{f_n\}_{n\geq 1}^{\infty} \subset M(X, \mathfrak{X}))$
Then $\sup_n f_n(x)$, $\inf_n f_n(x)$, $\limsup_{n\to\infty} f_n(x)$, $\liminf_{n\to\infty} f_n(x)$ are measurable functions in $M(X,\mathfrak{X})$

Proof $g(x) = \sup_n f_n(x)$. For any $c \in \mathbb{R}$

$$\{x \in X : g(x) > c\} = \bigcup_{n=1}^{\infty} \{x \in X : f_n(x) > c\}$$

$\Rightarrow$ $g(x)$ is measurable.

$h(x) = \inf_n f_n(x)$. For any $c \in \mathbb{R}$

$$\{x \in X : h(x) \leq c\} = \bigcap_{n=1}^{\infty} \{x \in X : f_n(x) \leq c\}$$

$\Rightarrow$ $h(x)$ is measurable.

$$\limsup_{n\to\infty} f_n(x) = \inf_k \sup_{n\geq k} f_n(x) \Rightarrow \text{ok}$$

$$\liminf_{n\to\infty} f_n(x) = \sup_k \inf_{n\geq k} f_n(x) \Rightarrow \text{ok} \quad \blacksquare$$

Note that if $\{f_n\} \subset M(X, \mathfrak{X})$ and $f_n(x) \to f(x)$ pointwise then $f \in M(X, \mathfrak{X})$

Def A simple function is a finite linear combination of characteristic functions of measurable sets.

It's clear that simple function is measurable

$$f(x) = \sum_{n=1}^{N} a_i \chi(A_i)$$

3

Lemma Let $f \in M(X, \underline{X})$, $f \geq 0$. Then $\exists \{\phi_n\} \subset M(X, \underline{X})$:

1) $0 \leq \phi_n(x) \leq \phi_{n+1}(x) \quad \forall x \in X, n \in \mathbb{N}$
2) $\lim_{n \to \infty} \phi_n(x) = f(x)$
3) ϕ_n is a simple function

Proof Let us fix $n \in \mathbb{N}$ and define sets $E_{k,n} = \{x : f(x) \in [\frac{k}{2^n}, \frac{k+1}{2^n})\}$ for $0 \leq k \leq n2^{n} - 1$ $(k \in \mathbb{N})$ and $E_{n2^n, n} = \{x : f(x) \in [n, \infty)\}$

In this way we split the range of $f(x)$ and base $f(x)$. It is clear that $\mathbb{R}_+ = \bigcup_{k=0}^{n2^{n}-1} [\frac{k}{2^n}, \frac{k+1}{2^n}) \cup [n, \infty)$ and $X = \bigcup_{k=0}^{n2^n} E_{k,n}$.

Moreover, it is also clear that $E_{k,n}$ are disjoint and measurable.

We define $\phi_n(x) = \frac{k}{2^n}$ if $x \in E_{k,n}$, i.e.

$$\phi_n(x) = \sum_{k=0}^{2^n n} \frac{k}{2^n} \chi_{E_{k,n}}(x)$$

It is clear that ϕ_n is a simple function.

We have $\phi_{n+1}(x) = \sum_{k=0}^{2^{n+1}(n+1)} \frac{k}{2^{n+1}} \chi_{E_{k,n+1}}(x)$

Take $x \in E_{k,n+1}$ then $\phi_{n+1}(x) = \frac{k}{2^{n+1}}$

$\phi_n(x) \leq \phi_{n+1}(x)$ as partition for $n+1$ is finer.

It is also clear that $\phi_n(x) \longrightarrow f(x)$ $\square$

1

Lecture 5

Def A function $f: X \to \mathbb{R}$ is called elementary if it is measurable & takes no more than a countable # of values

It is clear that elementary function taking values $y_1, y_2 \ldots$ is measurable iff $A_n = \{x \in X : f(x) = y_n\}$ are measurable

Lemma A function $f: X \to \mathbb{R}$ is measurable iff it is a limit of uniformely convergent sequence of elementary functions.

Proof Let $\{f_n(x)\}$ be a sequence of elementary functions and $f_n \rightrightarrows f$ on X.
As convergence is uniform $\Rightarrow$ it is pointwise $\Rightarrow$
$\Rightarrow$ by a previous result we are done.
Now let f be a measurable function
we define

$$f_n(x) = \frac{m}{n} \quad \text{on } A_n^m = \left\{x \in X : \frac{m}{n} \le f(x) < \frac{m+1}{n}\right\}$$

$m \in \mathbb{Z}$ & $n \in \mathbb{N}$

Obviously $\{f_n\}$ are elementary and

$$|f_n(x) - f(x)| \le \frac{1}{n} \quad \text{on } X \quad \Rightarrow \text{done}$$

②

Measure

Def Let $(X, \mathcal{X})$ be a measurable space. Then $\mu : \mathcal{X} \to \mathbb{R}_+ \cup \{+\infty\}$ is a measure if

1) $\mu(\emptyset) = 0$
2) $\forall A \in \mathcal{X}$ $\mu(A) \geq 0$
3) if $\{A_n\}_{n \geq 1}^{\infty} \subset \mathcal{X}$ and are disjoint then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

$(X, \mathcal{X}, \mu)$ is called a measure space.

Def A measure μ is called a finite measure if $\mu(X) < \infty$; μ is a σ-finite measure if $\exists \{A_n\}_{n \geq 1}^{\infty}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty \; \forall n$.

Note that $\exists!$ measure on $(\mathbb{R}, \mathcal{B})$ such that $\mu((a,b)) = b - a$. This is Lebesgue measure.

Lemma (Monotonicity) Let μ be a measure on $(X, \mathcal{X})$. If $A, B \in \mathcal{X}$ and $A \subset B$ then $\mu(A) \leq \mu(B)$. Moreover if $\mu(A) < \infty$ then

$$\mu(B \setminus A) = \mu(B) - \mu(A)$$

Proof It is clear that $B = A \cup (B \setminus A)$

So $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$ ∎

③

Lemma Let μ be a measure $(X, \underline{X})$

1) if $\{A_n\}_{n=1}^{\infty}$ is increasing sequence $(A_1 \subset A_2 \subset \cdots)$ of measurable sets in $\underline{X}$ then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$

2) if $\{B_n\}_{n=1}^{\infty}$ is decreasing sequence $(B_1 \supset B_2 \supset \cdots)$ of measurable sets in $\underline{X}$ $(\mu(B_1) < \infty)$ then

$$\mu\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n)$$

Proof Take $A_0 = \phi$ $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n \setminus A_{n-1}$

$A_n \setminus A_{n-1}$ are disjoint.

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n \setminus A_{n-1}) =$$

$$= \sum_{n=1}^{\infty} \mu(A_n) - \mu(A_{n-1}) = \lim_{n\to\infty} \mu(A_n)$$

Def A measure space $(X, \underline{X}, \mu)$ is complete if $A \subset N \in \underline{X}$ and $\mu(N) = 0$ implies $A \in \underline{X}$.

Def Let $(X, \underline{X}, \mu)$ be a measure space. We say that proposition is true a.e. if it hold on $X \setminus N$ with $\mu(N) = 0$.

Proof

$$\int_X f\,d\mu = \sup_{\substack{\varphi \le f \\ \varphi \text{ is simple}}} \int_X \varphi\,d\mu$$

Fix $\alpha \in (0,1)$ & Fix φ simple : $\varphi \le f$ and define

$$A_n = \{x \in X : \alpha\varphi(x) \le f_n(x)\}$$. Since $f_n \le f_{n+1}$

$A_1 \subset A_2 \subset A_3 \dots$

Moreover $\bigcup_{n\ge1}^{\infty} A_n = X$

We now define $\nu(A) = \int_A \varphi\,d\mu$

By previous results ν is a measure and hence

$$\nu\left(\bigcup_{n\ge1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n)$$

$$\nu(X) = \int_X \varphi\,d\mu$$

Hence $\int_X \varphi\,d\mu = \lim_{n\to\infty} \nu(A_n)$

On the other hand

$$\int_X f_n\,d\mu \ge \int_{A_n} f_n\,d\mu \ge \alpha \int_{A_n} \varphi(x)\,d\mu = \alpha\,\nu(A_n)$$

$$\lim_{n\to\infty} \int_X f_n\,d\mu \ge \alpha \lim_{n\to\infty} \nu(A_n) = \alpha \int_X \varphi\,d\mu$$

Take sup over all $\varphi \le f$ we obtain

$$\lim_{n\to\infty} \int_X f_n\,d\mu \ge \alpha \int_X f\,d\mu \qquad \forall \alpha \in (0,1)$$

Take a limit as $\alpha \to 1$ & done ∎

Corollary $f, g \in M^+(X, \underline{X})$, $\alpha, \beta \in \mathbb{R}^+ \Rightarrow$
$\Rightarrow \alpha f + \beta g \in M^+(X, \underline{X})$ and

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu$$

Proof Take φ_n - simple, $\varphi_n \le \varphi_{n+1}$ and
$\varphi_n \to f$, ψ_n - simple $\psi_n \le \psi_{n+1}$ &
$\psi_n \to g$

Then $\alpha\varphi_n + \beta\psi_n \longrightarrow \alpha f + \beta g$ $\quad \alpha\varphi_n + \beta\psi_n \le \alpha\varphi_{n+1} + \beta\psi_{n+1}$

$$\begin{array}{ccccc} \displaystyle\int_X (\alpha\varphi_n + \beta\psi_n)\, d\mu & = & \alpha \displaystyle\int_X \varphi_n\, d\mu & + & \beta \displaystyle\int_X \psi_n\, d\mu \\ \downarrow & & \downarrow & & \downarrow \\ \displaystyle\int_X (\alpha f + \beta g)\, d\mu & & \alpha \displaystyle\int_X f\, d\mu & + & \beta \displaystyle\int_X g\, d\mu \end{array}$$

■

Lemma (Fatou) Let $(X, \underline{X}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of functions in $M^+(X, \underline{X})$. Then

$$\liminf_{n\to\infty} \int_X f_n\, d\mu \ge \int_X \liminf_{n\to\infty} f_n$$

Proof We define $\varphi_n(x) = \inf_{k \ge n} f_k(x)$

It is clear that $\varphi_n(x) \le \varphi_{n+1}(x)$.

Moreover $\lim_{n\to\infty} \varphi_n(x) = \liminf_{n\to\infty} f_n(x)$

By monotone convergence

$$\lim_{n\to\infty} \int_X \varphi_n(x)\, d\mu = \int_X \liminf_{n\to\infty} f_n(x)\, d\mu$$

$$\liminf \int_X \varphi_n\, d\mu \le \liminf \int_X f_n\, d\mu$$

■

Lemma. Let $(X, \underline{X}, \mu)$ be a measure space and $\{f_n\} \subset M^+(X, \underline{X})$. Then

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

Proof Trivial.

Lemma Let $(X, \underline{X}, \mu)$ be a measure space and $f \in M^+(X, \underline{X})$. Define

$$\nu(A) = \int_A f \, d\mu \quad \forall A \in \underline{X}.$$

Then ν is a measure.

Proof Trivial.

Def Let $(X, \underline{X}, \mu)$ be a measure space. A statement holds a.e. if $\exists N \in \underline{X}$ such that $\mu(N) = 0$ & statement holds on $X \setminus N$.

Def A measure ~~[illegible]~~ space $(X, \underline{X}, \mu)$ is complete if $A \subset N \in \underline{X}$ and $\mu(N) = 0$ implies $\mu(A) = 0$ and $A \in \underline{X}$.

Theorem Let $(X, \underline{X}, \mu)$ be a measure space and let $f \in M^+$. Then $f(x) = 0$ a.e. $\iff \int_X f \, d\mu = 0$

Lecture 9

Lemma (Chebyshev) Let $f \in M^+(X, \underline{X})$ and $c > 0$. Then

$$\mu(\{x \in X : f(x) \geq c\}) \leq \frac{1}{c} \int_X f \, d\mu$$

Proof Take $B = \{x \in X : f(x) \geq c\}$

$$\int_X f \, d\mu \geq \int_B f \, d\mu \geq c \int_B d\mu = c\,\mu(B) \quad \blacksquare$$

Def Let $(X, \underline{X}, \mu)$ be a measure space. A statement holds a.e if $\exists\, N \in \underline{X}$: $\mu(N) = 0$ & statement holds on $X \setminus N$.

Def Let $\{f_n\} \subset M(X, \underline{X})$ and $(X, \underline{X}, \mu)$ be a measure space. We say that

$$f_n(x) \longrightarrow f(x) \quad \text{a.e } x \in X$$

if $\mu(\{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0$

Def A measure space $(X, \underline{X}, \mu)$ is complete if $A \subset N \in \underline{X}$ and $\mu(N) = 0$ implies $A \in \underline{X}$.

Lemma $\{f_n\} \subset M(X, \underline{X})$ & $f_n \longrightarrow f$ a.e then $f \in M(X, \underline{X})$.

__Lemma__ Let $(X, \underline{X}, \mu)$ be a measure space, $f \in M^+(X, \underline{X})$. Then

$f(x) = 0$ μ a.e on $X \iff \int f d\mu = 0$

__Proof__ $f(x) = 0$ a.e $\Rightarrow$ $E = \{x \in X : f(x) > 0\}$ has $\mu(E) = 0$.

Define $f_n = n \cdot \chi_E$ it's clear $\int_X f_n d\mu = 0$

and $\liminf_{n \to \infty} f_n \geq f$

Using Fatou we get the result.

Assume $\int_X f d\mu = 0$ take

$$A_n = \{x \in X : f(x) \geq \tfrac{1}{n}\}$$

Using Chebyshev we have

$$\mu(A_n) \leq n \int_X f d\mu = 0 \qquad (A_n \subset A_{n+1})$$

But $A = \{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n$

$$\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0$$

__Theorem__ (MCT a.e)

Lecture 10

We consider extended $\bar{\mathbb{R}}$.

Thm Let $(X, \underline{X}, \mu)$ be a measure space.
Let $\{f_n\}_{n=1}^{\infty} \subset M^+(X, \underline{X})$, $f_n \leq f_{n+1}$
and $f_n \longrightarrow f$ a.e.
Then $\int_X f_n \, d\mu \longrightarrow \int_X f \, d\mu$

Proof Let $N = \{x \in X : f_n \not\to f\}$

Then $f_n \longrightarrow f$ on $X \setminus N$ and then

$$\lim_{n\to\infty} \int_{X\setminus N} f_n \, d\mu = \int_{X \setminus N} f \, d\mu$$

But $\int_N f_n \, d\mu = 0$ as $\int_X f_n \chi_N \, d\mu = 0$ $\left(\chi_N f_n \underset{a.e.}{=} 0\right)$

and $\int_N f \, d\mu = 0 \implies$ done ∎

Def. Let μ & ν be measures on $(X, \underline{X})$.
Then ν is absolutely continuous wrt μ, $\nu \ll \mu$,
if $A \in \underline{X}$ and $\mu(A) = 0 \implies \nu(A) = 0$

Lemma Let $f \in M^+(X, \underline{X})$; $\nu : \underline{X} \to \bar{\mathbb{R}}$

$\nu(A) = \int_A f \, d\mu$. Then $\nu \ll \mu$.

Proof Trivial.

Thm

Integrable functions

Def Let $f: X \to \mathbb{R}$ be a measurable function. We call f integrable iff

$$\int_X |f|\, d\mu < \infty$$

We define $\int_X f\, d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$

We denote all integrable functions as $L(X, \mathfrak{X}, \mu)$

Lemma $L(X, \mathfrak{X}, \mu)$ is a vector space over $\mathbb{R}$.

Proof $f, g \in L(X, \mathfrak{X}, \mu)$, $\alpha, \beta \in \mathbb{R}$

then $\int_X |\alpha f + \beta g| \le \alpha \int_X |f|\, d\mu + \beta \int_X |g|\, d\mu < C$

Theorem (LDCT) Let $\{f_n\} \subset M(X, \mathfrak{X})$,

$f_n \to f$ a.e. on X.

Assume $\exists\, g \in L(X, \mathfrak{X}, \mu) : |f_n| \le g \;\; \forall n \in \mathbb{N}$

Then $f \in L(X, \mathfrak{X}, \mu)$ &

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$$

Proof 1) $|f_n| \le g \Rightarrow |f| \le g \Rightarrow f \in L(X, \mathfrak{X}, \mu)$.

2) $f_n + g \ge 0 \Rightarrow$ use Fatou

$\liminf \int_X f_n + g \ge \int_X f + g \Rightarrow \lim \int_X f_n\, d\mu \ge \int_X f\, d\mu$

However

$g - h_n \geq 0$ as well

$\liminf \int_X (g - h_n)\, d\mu \geq \int_X g - f\, d\mu$

$\Rightarrow \limsup \int_X h_n\, d\mu \leq \int f\, d\mu$ ∎

<u>Lemma</u> $f \in L(X, \mathfrak{X}, \mu) \Rightarrow$

$$\mu(\{x \in X : |f(x)| = \infty\}) = 0$$

<u>Proof</u>

$$\mu(\{x \in X : |f(x)| \geq n\}) \leq \frac{1}{n} \int_X |f|\, d\mu \leq \frac{C}{n}$$

$$\{x \in X : |f(x)| = +\infty\} = \bigcap_{n=1}^{\infty} \{x \in X : |f(x)| \geq n\}$$

$$\Rightarrow \mu\left(\bigcap_{n=1}^{\infty} \{|f(x)| \geq n\}\right) = \lim \mu(A_n) = 0 \text{ ∎}$$

<u>Def</u> Let $(X, \mathfrak{X})$ be a measurable space.
Then $\nu : X \to \mathbb{R}$ is a charge if

1) $\nu(\emptyset) = 0$
2) $\{A_n\}_{n=1}^{\infty}$ are disjoint $\Rightarrow \nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$

<u>Lemma</u> Let $f \in (X, \mathfrak{X}, \mu)$ and define
$\nu : X \to \mathbb{R}$ as $\nu = \int_A f\, d\mu \Rightarrow \nu$ is a charge.

Lecture 11

Theorem Assume $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ $\forall x \in X$

and $\exists\, g \in L(X, \bar{X}, \mu)$: $|f(x,t)| \le g(x)$.

($f(x,t)$ is measurable).

Then $\int_X f(x,t_0)\,d\mu(x) = \lim_{t\to t_0} \int_X f(x,t)\,d\mu(x)$ $\boxed{}$

Proof Take $t_n \to t_0$ & $f_n(x) = f(t_n, x)$

Corollary Assume $f(x,t)$ is measurable and $t \longmapsto f(x,t)$ is continuous on $[a,b]$

for each $x \in X$

If $\exists\, g \in L(X, \bar{X}, \mu)$: $|f(x,t)| \le g(x)$ $\forall t$

Then $F(t) = \int_X f(x,t)\,d\mu(x)$ is continuous.

Theorem Assume $x \longmapsto f(x,t_0)$ is integrable

on X for some $t_0 \in [a,b]$. and is measurable

Assume $\frac{\partial f}{\partial t}$ exists on $X \times [a,b]$

$\left|\frac{\partial f}{\partial t}(x,t)\right| \le g(x)$ $\forall t \in [a,b]$

Then $\frac{\partial F}{\partial t} = \frac{\partial}{\partial t} \int_X f(x,t)\,d\mu(x) = \int_X \frac{\partial f}{\partial t}(x,t)\,d\mu(x)$

Proof ① $f(x,t)$ is integrable $\forall t \in [a,b]$

$|f(x,t)| \le |f(x,t_0)| + |f(x,t) - f(x,t_0)|$

$\frac{f(x,t) - f(x,t_0)}{t - t_0} = \frac{\partial f}{\partial t}(x, t^*) \Rightarrow$ done.

② Define $f_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}$ $\quad t_n \to t_0$

$f_n(x) \longrightarrow \frac{\partial f}{\partial t}(x,t)$ and $f_n(x) \in L(X, \bar{X}, \mu)$

$|f_n(x)| \le g(x)$

By DCT we have

$$\int_X h_n \, d\mu(x) \to \int_X \frac{\partial f}{\partial t}(x,t)\, d\mu(x)$$

$\Downarrow$

$$\frac{\partial}{\partial t} \int_X f(x,t)\, d\mu \quad \blacksquare$$

<u>Theorem</u> Assume $t \longmapsto f(x,t)$ is continuous on $[a,b]$ $\forall x \in X$ and $\exists g \in L(X, \mathfrak{X}, \mu)$: $|f(x,t)| \le g(x)$ $\forall t \in [a,b]$ (Measurable)

Then $$\int_a^b \int_X f(x,t)\, d\mu \, dt = \int_X \int_a^b f(x,t)\, dt\, d\mu$$

Integral in t is Riemann.

<u>Proof</u> Define $h(x,t) = \int_a^t f(x,s)\, ds$

We have ① $\frac{\partial h}{\partial t}(x,t) = f(x,t)$

② $h(x,t)$ is measurable $\forall t$

③ $h(x,t)$ is integrable $\forall t$

We define $$H(t) = \int_X \left(\int_a^t f(x,s)\, ds \right) d\mu(x)$$

$$\frac{dH}{dt} = \int_X \frac{\partial h}{\partial t}(x,t)\, d\mu(x) = \int_X f(x,t)\, d\mu(x) = F(t)$$

Hence $$\int_a^b F(t)\, dt = H(b) - H(a) = \int_X \int_a^b f(x,s)\, ds\, d\mu \quad \blacksquare$$

Lecture 12

We have elementary functions $f: X \to R$

$$f(x) = \sum_{n=1}^{\infty} y_n \chi_{A_n}(x) \qquad A_n \text{ are disjoint}$$

and result
Any $f: X \to R$ is measurable iff it is a limit of uniformly convergent sequence of elementary functions.

Integral for elementary function is

$$(*) \quad \int_X f \, d\mu = \sum_{n=1}^{\infty} y_n \mu(A_n)$$

Def An elementary function $f: X \to R$ is integrable if series $(*)$ is absolutely convergent (i.e. $\sum_{n=1}^{\infty} |y_n| \mu(A_n) < \infty$)

Def A function $f: X \to R$ is integrable on $A \in \mathfrak{X}$ ($\mu(A) < \infty$) if $\exists \{f_n\}$ – elementary integrable functions $f_n \rightrightarrows f$ on A.

$$\int_A f \, d\mu = \lim_{n \to \infty} \int_A f_n \, d\mu$$

1) if $\{f_n\} \rightrightarrows f$ uniformly $\Rightarrow$
$\Rightarrow \int_A f_n \, d\mu$ converges to some α

2) if $\{f_n\}$ & $\{g_n\} \rightrightarrows f \Rightarrow \left| \int f_n \, d\mu - \int g_n \, d\mu \right| \to 0$

① Lecture

1) L_p inclusions :

— If $\mu(X) < \infty$ & $p > q \geq 1$ then $L^p(X) \subset L^q(X)$
Proof by Hölder.

— If $\mu(X) = \infty$ then above result is not true
Counter example $X = [1, \infty)$ $f(x) = \frac{1}{x}$ $f \in L^2$
$f \notin L^1$

— If $\mu(X) < \infty$ then $L^p \not\subset L^q$ for $1 \leq p < q$
$p = 1$ $q = 2$ $f(x) = \frac{1}{\sqrt{x}}$

2) Modes of convergence
Assume $\mu(X) < \infty$

Uniform convergence

$\swarrow$ $\downarrow$ $\searrow$

L^p convergence $\quad$ μ-c $\quad$ a.e C

$\downarrow$ $\qquad\qquad\qquad$ $\downarrow$

μ-c $\qquad\qquad\qquad$ μ-c

L^p convergence $\not\Rightarrow$ a.e convergence

<u>Example</u> $f_1 = \chi_{[0,\frac{1}{2}]}$ $f_2 = \chi_{[\frac{1}{2},1]}$, $f_3 = \chi_{[0,\frac{1}{3}]}$

$f_4 = \chi_{[\frac{1}{3},\frac{2}{3}]}$, $f_5 = \chi_{[\frac{2}{3},1]}$, etc

It is clear that $\int_{X=[0,1]} |f_n| \to 0$

but $f_n \not\to 0$ a.e
as for any $x \in [0,1]$ $\exists$ $f_{n_k}(x) \Rightarrow 1$

Therefore μ-convergence $\not\Rightarrow$ a.e convergence

2

Now a.e convergence $\not\Rightarrow$ L^p convergence

Take $f_n = \begin{cases} n & \text{on } [0, \frac{1}{n}] \\ 0 & \text{outside} \end{cases}$ $\quad X = (0,1)$

$f_n(x) \to 0$ for all $x \neq 0$

$\int_X |f_n| \to 1$

As we recall from DCT we need $|f_n| \leq g \in L_1$

Do we have a.e convergence $\to$ μ-convergence

Let us introduce another convergence

<u>Def</u> A sequence $\{f_n\}$, $f_n \to f$ almost uniformly if
$\forall \delta > 0 \ \exists E_\delta \in X : \mu(E_\delta) < \delta$
and $f_n \rightrightarrows f$ on $X \setminus E_\delta$

<u>Lemma</u> Almost uniform convergence implies a.e convergence & μ-convergence

<u>Proof</u> 1 AU $\to$ AE

Take F_m : $\mu(F_m) < \frac{1}{m}$ and $f_n \rightrightarrows f$ on F_m^c

Define $F = \bigcap_{m \geq 1} F_m$, clearly $\mu(F) = 0$

Take $x \in F^c \Rightarrow x \in F_m^c$ for some m

We know that $f_n \rightrightarrows f$ on $F_m^c \Rightarrow f_n(x) \to f(x)$

So $f_n(x) \to f(x) \ \forall x \in F^c \Rightarrow$ a.e

2 AU $\to$ μC

$\mu C \Leftrightarrow \forall a > 0 \lim_{n \to \infty} \mu(\{x \mid |f_n - f| > a\}) = 0$

(3) Fix any a & $\varepsilon > 0$

$\exists A_\varepsilon \in \underline{X}$: $f_n \rightrightarrows f$ on A_ε^c & $\mu(A_\varepsilon) < \varepsilon$

Now we can find $N > 0$ such that for $n \geq N$ $\{x \in X : |f_n(x) - f(x)| \geq a\} \subset A_\varepsilon$

$\Rightarrow \mu(\{x \in X : |f_n(x) - f(x)| \geq a\}) < \varepsilon$ ∎

<u>Theorem</u> (Egoroff) $f_n \to f$ a.e $\Rightarrow f_n \rightrightarrows f$ AU.
$(\mu(X) < \infty)$

<u>Proof</u> Later

<u>sub Ae</u> $\Leftarrow \mu c$

<u>Proof</u> ~~[illegible]~~

$\forall k \in \mathbb{N} \ \exists \ N(k) : n \geq N(k)$

$$\mu(\{x : |f_n(x) - f(x)| \geq \tfrac{1}{k}) \leq \tfrac{1}{2^k}$$

We can pick $N(1) < N(2) < \dots$
and thus have a subsequence $f_{n_k} = g_k$
such that for $n_k \geq N(k)$

$$\mu(\{x : |f_{n_k} - f(x)| \geq \tfrac{1}{k}\}) \leq \tfrac{1}{2^k}$$

$$n_1 < n_2 < n_3 \dots$$

Define $E_k = \{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}$

$H_m = \bigcup_{k \geq m}^{\infty} E_k$

$\mu(E_k) \leq \frac{1}{2^k} \Rightarrow \mu(H_m) \leq \frac{1}{2^{m-1}}$

$H = \bigcap_{m=1}^{\infty} H_m$ ~~m~~ $\mu(H) = 0$

if $x \notin H \Rightarrow x \notin H_m$ for some $m \Rightarrow x \notin E_k \ \forall k \geq m$
$\Rightarrow |f_{n_k}(x) - f(x)| \leq \frac{1}{k} \ \forall k \geq m$ ∎

$f_n \to f$ a.e. $\Rightarrow$ $f_n \to f$ in μ

$A = \{x : \lim_{n\to\infty} f_n(x) \neq f(x)\}$

$E_k(\varepsilon) = \{x : |f_k(x) - f(x)| \geq \varepsilon\}$

$R_n(\varepsilon) = \bigcup_{k\geq n}^{\infty} E_k(\varepsilon)$ $\quad M = \bigcap_{n\geq 1}^{\infty} R_n(\varepsilon)$

$\mu(R_n) \to \mu(M)$

We want to show $M \subset A$

if $x_0 \notin A \Rightarrow \lim_{n\to\infty} f_n(x_0) = f(x_0)$

$\Rightarrow \forall \varepsilon \; \exists \; n : |f_k(x_0) - f(x_0)| < \varepsilon \quad \forall k \geq n$

$\Rightarrow x_0 \notin R_n(\varepsilon) \Rightarrow x_0 \notin M$

$\Rightarrow M \subset A \Rightarrow \mu(M) = 0$ & $\mu(R_n) \to 0$

But $E_n(\varepsilon) \subset R_n(\varepsilon) \Rightarrow$

$\mu(E_n) \to 0 \Rightarrow$ done.

① Absolute continuity of LI

Theorem Let $f \in L$ then $\forall \varepsilon > 0 \ \exists \delta > 0$: if $\mu(A) < \delta$ $\int\limits_A |f| d\mu < \varepsilon$

Proof $\int\limits_{\{|f| > \lambda\}} |f| d\mu \underset{\lambda \to \infty}{\longrightarrow} 0 \Rightarrow$

$\Rightarrow \forall \varepsilon > 0 \ \exists \lambda$: $\int\limits_{\{|f| > \lambda\}} |f| < \frac{\varepsilon}{2}$

Now we take $\delta < \frac{\varepsilon}{2\lambda}$ and

$A \in \underline{X}$: $\mu(A) < \delta$

$$\int\limits_A |f| d\mu = \int\limits_{A \cap \{|f| > \lambda\}} |f| d\mu + \int\limits_{A \cap \{|f| \leq \lambda\}} |f| d\mu <$$

$$< \frac{\varepsilon}{2} + \lambda \mu(A) = \varepsilon$$

Theorem (Egoroff) Let $f_n \to f$ a.e. in $X \Rightarrow$

$\Rightarrow \forall \delta > 0$ $X_\delta \in \underline{X}$: $\mu(X \setminus X_\delta) < \delta$

and $f_n \rightrightarrows f$ on X_ε

Proof

Let $f_n \to f$ a.e. in X

Define $E_{n,m} = \bigcap\limits_{k \geq n} \{x \in X : |f_k(x) - f(x)| < \frac{1}{m}\}$

$$X_m = \bigcup_{n=1}^{\infty} E_{n,m}$$

$E_{n,m}$ are monotone $(E_{1,m} \subset E_{2,m} \subset \dots)$

We also see that on $E_{n,m}$ the difference $|f_k(x) - f(x)|$ is uniformly bounded $\forall k \geq n$

$\mu(E_{n,m}) \underset{n \to \infty}{\longrightarrow} \mu(X_m)$ by continuity of measure

$\forall m\ \exists n_0(m)$:

Hence $\mu(X^m \setminus E_{n_0(m),m}) < \frac{\delta}{2^m}$

We define $X_\delta = \bigcap_{m=1}^{\infty} E_{n_0(m),m}$

We claim X_δ is the set

1) $f_n \rightrightarrows f$ on X_ε

if $x \in X_\delta \Rightarrow x \in E_{n_0(m),m}\ \forall m \Rightarrow$

$\Rightarrow |f_k(x) - f(x)| < \frac{1}{m} \quad \forall k \geq n_0(m)$

2) $\mu(X \setminus X_m) = 0 \quad \forall m$

take $x \in X \setminus X_m \Rightarrow f_k(x) \not\to f(x)$

$\mu(X \setminus X_\delta) = \mu(X \setminus \bigcap_{m=1}^{\infty} E_{n_0(m),m}) =$

$= \mu(\bigcup_{m=1}^{\infty} (X \setminus E_{n_0(m),m})) \leq$

$\leq \sum \mu(X \setminus E_{n_0(m),m}) = \sum \mu(X^m \setminus E_{n_0(m),m}) < \sum \frac{\delta}{2^m} = \delta$

1

<u>Def</u> Two measures λ & μ are mutually singular if $\exists A \& B \in \underline{X}$: $A \cup B = X$, $A \cap B = \emptyset$

$\lambda(A) = 0$ & $\mu(B) = 0$.

We denote $\lambda \perp \mu$.

Recall Hahn decomposition:

$\nu : \underline{X} \to \mathbb{R}$ is a charge. $\exists$ positive $P \in \underline{X}$ and negative $N = X \setminus P$.

It is clear that $\nu^+ = \nu(P \cap A)$ & $\nu^- = -\nu(N \cap A)$ are mutually singular $\nu^+ \perp \nu^-$.

It is also clear that

$$\nu(A) = \nu(A \cap X) = \nu(A \cap P) + \nu(A \cap N) =$$
$$= \nu^+(A) - \nu^-(A)$$

<u>Theorem</u>. (Jordan decomposition)

Every charge ν has a unique decomposition into a difference $\nu = \nu^+ - \nu^-$ of two finite measures ν^+ & ν^- such that $\nu^+ \perp \nu^-$.

<u>Proof</u> Existence is trivial. $\Rightarrow$ $\nu = \nu^+ - \nu^-$, $P \cup N = X$

Assume $\nu = \mu^+ - \mu^-$, $\mu^+ \perp \mu^-$

Since $\mu^+ \perp \mu^-$ we can find $A \& B \in \underline{X}$:

$X = A \cup B$ and $\mu^+(A) = 0$ $\mu^-(B) = 0$

Take $E \in \underline{X}$

$$\nu(E \cap A) = \mu^+(E \cap A) - \mu^-(E \cap A) =$$
$$= -\mu^-(E \cap A) \le 0 \Rightarrow A \text{ is negative set}$$

(2)

$\nu(E\cap B) = \mu^+(E\cap B) - \mu^-(E\cap B) = \mu^+(E\cap B) \geq 0$

$\Rightarrow$ B is positive set

Therefore A, B is another Hahn decomposition of ν.

But we already know that

$\forall E \in \mathcal{X}$

$\nu^+(E) = \nu(E\cap P) = \nu(E\cap B) = \mu^+(E\cap B) = \mu^+(E\cap B) + \overset{0}{\overset{\shortparallel}{\mu^+(E\cap A)}} = \mu^+(E)$

$\nu^-(E) = -\nu(E\cap N) = -\nu(E\cap A) =$

$= \mu^-(E\cap A) = \mu^-(E\cap A) + \mu^-(E\cap B) = \mu^-(E)$

Hence $\nu^+ \equiv \mu^+$ & $\nu^- \equiv \mu^-$. ∎

__Corollary__ Let ν be a charge.

If $\nu = \nu_1 - \nu_2$ $\quad$ ν_1, ν_2 are finite measures

Then $\nu_1(A) \geq \nu^+(A)$ & $\nu_2(A) \geq \nu^-(A)$ $\forall A \in \mathcal{X}$.

__Proof__ Let $X = P \cup N$, $\nu^+(A) = \nu(A\cap P)$, $\nu^-(A) = -\nu(A\cap N)$

$\nu^+(A) = \nu(A\cap P) = \nu_1(A\cap P) - \nu_2(A\cap P) \leq \nu_1(A\cap P) \leq \nu_1(A)$

$\nu^-(A) = -\nu(A\cap N) = \nu_2(A\cap N) - \nu_1(A\cap N) \leq \nu_2(A\cap N) \leq \nu_2(A)$. ∎

We showed that Jordan decomposition is the __minimal__ ~~one~~ decomposition of ν into a difference of two ~~[illegible]~~ finite measures.

In fact $\nu^+(A) = \sup_{B\subset A} \nu(B)$

$\nu^-(A) = -\inf_{B\subset A} \nu(B)$

$\forall B \subset A$

$\nu(B) = \nu^+(B) - \nu^-(B) \leq \nu^+(B) \leq \nu^+(A) = \nu(A\cap P)$

$-\nu(B) = \nu^-(B) - \nu^+(B) \leq \nu^-(B) \leq \nu^-(A) = -\nu(A\cap N)$

Lecture

Theorem (Radon - Nikodym)
Let μ be a σ-additive finite measure on $(X, \underline{X})$
and ν be a signed measure on $(X, \underline{X})$,
$\nu \ll \mu$. There exists unique $f \in L^1(X, \mu)$:

$$\nu(A) = \int_A f \, d\mu$$

Proof We already know that
$\nu = \nu^+ - \nu^-$ and want to show if $\nu \ll \mu$
then $\nu^+ \ll \mu$ & $\nu^- \ll \mu$.
Let $E \in \underline{X}$ & $\mu(E) = 0 \Rightarrow \nu(E) = 0$.
But we also know that $\mu(E \cap P) = 0 \Rightarrow \nu(E \cap P) = 0 \Rightarrow$
$\Rightarrow \nu^+(E) = 0$
By the same arguments $\nu^-(E) = 0$.
Therefore we can now reduce the result for
finite measures rather than charges.
Assume ν is a finite measure,
we define the following set

$$K = \{ f \in L(X) ; f(x) \geq 0 , \int_A f(x) \, d\mu \leq \nu(A) \ \forall A \in \underline{X} \}$$

We can also define

$$M = \sup_{f \in K} \int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu$$

(It is clear $M < \infty$)

We define $g_n(x) = \max \{ f_1(x), \dots, f_n(x) \}$.
It is clear that $g_n \in K$:

1) $g_n \in L(X)$, $g_n \geq 0$ & have to show $\int_{E} g_n d\mu \leq \nu(E) \, \forall E$

2) $\exists \{E_i\}_{i=1}^{n}$ disjoint & such that $E = \bigcup_{i=1}^{n} E_i$ &

$$g_n(x) = f_i(x) \quad \text{on } E_i$$

Take $E_1 = \{x \in E : g_n(x) = f_1(x)\}$

$E_2 = \{x \in E \setminus E_1 : g_n(x) = f_2(x)\}$ etc

$$\int_E g_n d\mu = \sum_{i=1}^{n} \int_{E_i} f_i d\mu \leq \sum_{i=1}^{n} \nu(E_i) = \nu(E)$$

Now we define

$$f(x) = \sup_n f_n(x) = \lim_{n\to\infty} g_n(x).$$

By MCT we have $g_n \nearrow f$ and

$$\nu(E) \geq \int_E g_n d\mu \to \int_E f d\mu \implies f \in L(X,\mu)$$

we define

$$\lambda(E) = \nu(E) - \int_E f d\mu \geq 0$$

It's clear that λ is a finite measure.

We want to show $\lambda(E) = 0 \; \forall E \in \underline{X} \Rightarrow \lambda \equiv 0$

<u>Lemma</u> Let λ, μ be non-trivial measures and $\lambda \ll \mu$

Then $\exists \, n \in \mathbb{N}$ & $B \in \underline{X}$ such that

$\mu(B) > 0$ & B is positive wrt $\lambda - \frac{1}{n}\mu$.

<u>Proof</u> Let $X = A_n^- \cup A_n^+$ be Hahn decomposition of a signed measure $\lambda - \frac{1}{n}\mu$.

Define $A_0^- = \bigcap_{n=1}^{\infty} A_n^-$ & $A_0^+ = \bigcup_{n=1}^{\infty} A_n^+$. Then

$A_0^- \cup A_0^+ = X$. For any $n \in \mathbb{N}$ we have

$$\lambda(A_0^-) - \frac{1}{n}\mu(A_0^-) \leq 0 \implies \lambda(A_0^-) = 0$$

Since λ is a measure we have $(X = A_0^- \cup A_0^+)$

$\lambda(A_0^+) > 0$ (otherwise $\lambda \equiv 0$)

Therefore $\mu(A_0^+) > 0$ since if $\mu(A_0^+) = 0 \Rightarrow \lambda(A_0) = 0$ as $\lambda \ll \mu$.

Hence $\exists n : \mu(A_n^+) > 0$ (otherwise $\mu(A_0^+) = 0$)

and A_n^+ is a positive set for

$\lambda - \frac{1}{n}\mu$

We know that $\lambda = \nu - \int f \, d\mu$ satisfies

$\lambda \ll \mu$. By Lemma $\exists B$ & $n \in \mathbb{N}$

such that $\lambda(E \cap B) \geq \frac{1}{n}\mu(E \cap B)$

$\forall E \in \mathfrak{X}$ & $\mu(B) > 0$.

We define $h(x) = f(x) + \frac{1}{n}\chi_B(x)$

$$\int_E h(x) \, d\mu = \int_E f(x) d\mu + \frac{1}{n}\mu(E \cap B) \leq$$

$$\leq \int_E f \, d\mu + \lambda(E \cap B) =$$

$$= \int_E f \, d\mu + \nu(E \cap B) - \int_{E \cap B} f \, d\mu =$$

$$= \int_{E \setminus B} f \, d\mu + \nu(E \cap B) \leq \nu(E \setminus B) + \nu(E \cap B) = \nu(E)$$

Therefore $h \in K$ and

$$\int_X h \, d\mu = \int_X f \, d\mu + \frac{1}{n}\mu(B) > M \Rightarrow \text{contradiction}$$

$\Rightarrow \lambda \equiv 0$ ∎

<u>Uniqueness</u> if $\int_A f \, d\mu = \int_A g \, d\mu \;\; \forall A \Rightarrow$

$\Rightarrow \int_A f - g \, d\mu = 0 \;\; \forall A \Rightarrow \int_X |f-g| \, d\mu = 0 \Rightarrow f = g$ a.e

~~[illegible]~~

σ-finite measures case

Let ν & μ be σ-finite measures, then

$\exists \{A_n\}_{n=1}^{\infty}$: $\nu(A_n) < \infty$, $\mu(A_n) < \infty$

and $A_n \subset A_{n+1}$ $\bigcup_{n=1}^{\infty} A_n = X$

$\forall n$ we can find $f_n \in M^+$ such that

$\forall E \subset A_n$ $\nu(E) = \int_E f_n d\mu$ and we extended

f_n by 0 outside A_n.

It is clear that $f_n \le f_{n+1}$

as by uniqueness f_{n+1} coincides with f_n outside

and $f_{n+1} \geq 0$ ~~[illegible]~~ on $A_{n+1} \setminus A_n$.

Therefore $f_n \nearrow$ and it has a limit

$\lim_{n\to\infty} f_n(x) = f(x)$

Using MCT $\forall A \in X$ we have

$$\nu(A) = \lim_{n\to\infty} \lambda(A \cap A_n) = \lim_{n\to\infty} \int_A f_n d\mu =$$

$$= \int_A f d\mu$$

$\frac{d\nu}{d\mu} = f$ is Radon-Nikodym derivative

(2)

Theorem (Lebesgue decomposition)

Let λ, μ be σ-finite measures on $\mathfrak{X}$

$\exists$ measures λ_1 & λ_2 such that $\lambda_1 \perp \mu$ & $\lambda_2 \ll \mu$

and $\lambda = \lambda_1 + \lambda_2$. λ_1 & λ_2 are unique.

Proof Take $\nu = \lambda + \mu$. It's clear

that $\lambda \ll \nu$ & $\mu \ll \nu$

$\Rightarrow$ $\exists f, g \in M^+$ such that $\forall A \in \mathfrak{X}$

$$\mu(A) = \int_A f \, d\nu \qquad \lambda(A) = \int_A g \, d\nu$$

Take $B = \{x \in X : f(x) = 0\}$. Note $\mu(B) = 0$

Define λ_1 & λ_2 as follows

$$\lambda_1(A) = \lambda(A \cap B) \qquad \lambda_2(A) = \lambda(A \cap B^c)$$

It is clear that $\lambda_1(X \setminus B) = 0$ $\Rightarrow$

$\lambda_1 \perp \mu$

We now have to show $\lambda_2 \ll \mu$.

Take $A \in \mathfrak{X} : \mu(A) = 0$ $\Rightarrow$ $f(x) = 0$ ν-a.e $x \in A$

$\Rightarrow$ $f(x) = 0$ λ-a.e $x \in A$

Hence $\lambda_2(A) = \lambda(A \cap B^c) = 0$ $\Rightarrow$

$\Rightarrow$ $\lambda_2 \ll \mu$. ∎

Assume $\exists$ α_1 & α_2 $\lambda = \alpha_1 + \alpha_2$ $\alpha_1 \perp \mu$ & $\alpha_2 \ll \mu$

$$\alpha_1 + \alpha_2 = \lambda_1 + \lambda_2 \Rightarrow \underbrace{\alpha_1 - \lambda_1}_{\perp \mu} = \underbrace{\lambda_2 - \alpha_2}_{\ll \mu}$$

$\nu \perp \mu$ & $\nu \ll \mu$ $\Rightarrow$ $\nu = 0$ ∎