MICROMAGNETICS OF THIN SHELLS

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We study a thin-shell limit of micromagnetic energy for soft small ferromagnets. The relations between thickness of the magnet $t$, diameter $l$ and magnetic exchange length $w$ are $t/l \to 0$ and $tl/w^2 \lesssim 1$. We prove a $\Gamma$-convergence of the original 3D problem to a nonlocal 2D problem.

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1. Introduction

The micromagnetic variational principle is a non-convex, nonlocal variational problem whose local minima represent the stable magnetization patterns in a ferromagnetic body. The multi-scale complexity of the micromagnetic functional creates a lot of regimes, depending on the relation between the material and geometrical parameters. Our work explores one of these regimes, by considering specific one-parameter family of micromagnetic problems. We focus on a special case of nonuniform soft thin ferromagnetic films — thin shells, see Fig. 1.

After a suitable normalization, the micromagnetic energy has the form

$$\mathcal{E}(m) = w^2 \int_{\Omega} |\nabla m|^2 + Q \int_{\Omega} \phi(m) + \int_{\mathbb{R}^3} |\nabla u|^2 - 2 \int_{\Omega} h_{\text{ext}} \cdot m. \quad (1.1)$$

Here $\Omega \subset \mathbb{R}^3$ is the region occupied by the ferromagnet; $m: \Omega \to \mathbb{R}^3$ is the normalized magnetization, constrained by

$$|m(x)| = \begin{cases} 
1 & \text{for } x \in \Omega, \\
0 & \text{for } x \in \mathbb{R}^3 \setminus \Omega
\end{cases} \quad (1.2)$$

and $h_{\text{ext}}$ is the external applied field. The function $u$ is defined by

$$\text{div} (\nabla u + m) = 0 \quad \text{in } \mathbb{R}^3, \quad (1.3)$$
in the sense of distributions. The four different energy terms are called exchange, anisotropy, magnetostatic and Zeeman energies, respectively. For more information about micromagnetics see Refs. 1 and 6.

We observe that in soft ferromagnets \((Q = 0)\) under no applied field \((h_{\text{ext}} = 0)\) the nondimensionalized problem has two parameters:

\[
\begin{align*}
    h &= \frac{t}{l}, \text{ the normalized thickness of the sample;} \\
    d &= \frac{w}{l}, \text{ the normalized exchange length,}
\end{align*}
\]

where \(t, l\) and \(w\) are the thickness of the magnet, the diameter of the magnet and the exchange length of the ferromagnetic material, respectively. Relation between \(h\) and \(d\) creates a lot of different regimes.

This paper studies soft thin shells in the regime \(h \to 0\) and \(\frac{h}{d^2} \to \frac{1}{\alpha} \geq 0\). The thin shell domain under consideration is defined as

\[
\Omega_h = \{(x, z) : hf_2(x) \leq z \leq hf_1(x), \ x \in \omega \subset \mathbb{R}^2\},
\]

where functions \(f_1\) and \(f_2\) vanish on the boundary of \(\omega\), see Fig. 1. Note that it is different from a thin film domain \(V_h = \omega \times (0, h)\) (see Fig. 2) studied in various regimes in Refs. 2–5, 7 and 9. Numerical simulations\(^{10}\) suggest that magnetization behavior in thin shells differs from that in thin films, we confirm this statement by rigorous analysis. We treat the rescaled thickness \(h\) as a small parameter and explore the asymptotics of the magnetostatic energy with respect to \(h\). Then using
methods of $\Gamma$-convergence we prove that the full micromagnetic functional reduces to the following 2D nonlocal variational problem

$$\min \alpha \int_\omega |\nabla' m|^2 (f_1 - f_2) + \|\text{div} (m(f_1 - f_2))\|^2_{H^{-1/2}(\omega)}$$  \hspace{1cm} (1.4)$$

with constraints $m = m(x)$ for $x \in \omega$, $|m| = 1$ and $m_3 = 0$. If $\alpha = \infty$, there is a natural constraint $m = \text{const}$. The nonlocal $H^{-1/2}$ norm in (1.4) comes from the magnetostatic energy term. For simplicity we assumed $Q = 0$ and $h_{\text{ext}} = 0$ but it is easy to include appropriately scaled anisotropy and Zeeman term, since $\Gamma$-convergence is insensitive to compact perturbations of the functional.

It is interesting to compare limiting problem (1.4) to the ones obtained in the corresponding regime for thin films. Regime when $h \to 0$ and $h |\log h|^d \to \alpha > 0$ was studied in Ref. 7. The limiting variational problem in this case is

$$\min \alpha \int_\omega |\nabla' m|^2 + \frac{1}{2\pi} \int_{\partial \omega} (m \cdot n)^2$$  \hspace{1cm} (1.5)$$

with constraints $m = m(x)$ for $x \in \omega$, $|m| = 1$ and $m_3 = 0$. It is two-dimensional and local, the contribution of stray field energy reduces to a constant times the boundary integral of $(m \cdot n)^2$.

Kurzke$^8$ was investigating a reduced model similar to (1.4) and proved the formation of boundary vortices in the limit $\alpha \to 0$. To explain it formally we notice that magnetization $m$ in problem (1.5) has constraints $|m| = 1$, $m_3 = 0$ and as $\alpha \to 0$ it prefers to be aligned along the boundary of the magnet ($(m \cdot n) = 0$ on $\partial \omega$). Therefore it is forced to form a closed flux configuration and it turns out that boundary vortices are energetically preferable over interior vortices. The work of Kurzke formally corresponds to the regime $h \to 0$ and $h |\log h|^d \to \alpha > 0$. Recently, Moser$^9$ showed the formation of boundary vortices in the regime $h \to 0$ and $\frac{h}{|\log h|^d} \to \alpha > 0$ for the original micromagnetic functional (1.1).

In thin shells due to the fact that $f_i(\partial \omega) = 0$, in the limit we do not have a constraint $(m \cdot n) = 0$. This means that, unlike in a thin film, magnetization is not forced to form a vortex and, for instance, configuration $m = \text{const.}$ $|m| = 1$, $m_3 = 0$ has a finite energy.

The paper is organized as follows. In Sec. 2 we state the mathematical problem and give our main result which establishes the $\Gamma$-convergence of the micromagnetic energy to a suitable 2D variational problem. In Sec. 3 we prove $\Gamma$-convergence theorem relying on the asymptotic expansion of magnetostatic energy. In Sec. 4 we prove some auxiliary propositions used in Sec. 3, providing a simplification of a nonlocal term.

2. Statement of the Results

We consider one-parameter family of micromagnetic energy functionals

$$E_h(m_h) = \frac{d^2}{h^2} \int_{\Omega_h} |\nabla m_h|^2 + \frac{1}{h^2} \int_{\mathbb{R}^3} |\nabla u_h|^2,$$  \hspace{1cm} (2.1)$$
where
\[ \Omega_h = \{(x, z) : hf_2(x) \leq z \leq hf_1(x), \ x \in \omega \subset \mathbb{R}^2\}, \]
|\(m_h| = 1 \text{ and } u_h \text{ satisfies the following equation} \]
\[ -\Delta u_h = \text{div}\ (m_h \chi(\Omega_h)) \quad \text{in } \mathbb{R}^3. \]

**Hypothesis (H).** We assume the following properties of \(f_1\) and \(f_2\):

- \(f_1(x) \geq f_2(x) \) on \(\omega; \ f_i \in C_0^1(\omega); \)
- for any \(x, y \in \omega\) we have \(|f_1(x) - f_i(y)| \leq C|x - y|^\gamma\) for some constant \(C\) and \(0 < \gamma \leq 1; \)
- \(\int_\omega |\nabla f_i|^2 \leq C.\)

Rescaling domain in \(z\) direction, we obtain
\[ E_h(\tilde{m}_h) = \frac{d^2}{h} \int_\Omega \left( |\nabla' \tilde{m}_h|^2 + \frac{1}{h^2} \left( \frac{\partial \tilde{m}_h}{\partial z} \right)^2 \right) + \frac{1}{h^2} \int_{\mathbb{R}^3} |\nabla u_h|^2, \tag{2.2} \]
where \(\tilde{m}_h(x, z) = m_h(x, hz)\) for \(x = (x_1, x_2) \in \omega, \ z \in (f_1(x), f_2(x)), \ \nabla' = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)\) and
\[ \Omega = \{(x, z) : f_1(x) \leq z \leq f_2(x), \ x \in \omega \subset \mathbb{R}^2\}. \]

Note that magnetostatic energy is written as in (2.1). This is the functional we are going to consider below. Throughout the paper we use the following notation: \(x, y\) are points in \(\mathbb{R}^3; x, y\) are points in \(\omega \subset \mathbb{R}^2; \ s, t \in \mathbb{R}\) are thickness variables:
\[ \|a\|^2_{H^{-1/2}(\omega)} = \frac{1}{4\pi} \int_\omega \int_\omega \frac{a(x)a(y)}{|x - y|}, \tag{2.3} \]
\[ \Gamma_h^1(x, y) = \frac{1}{h} \left( \frac{1}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_1(y))^2}} - \frac{1}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}} \right), \tag{2.4} \]
\[ \Gamma_h^2(x, y) = \frac{1}{h} \left( \frac{1}{\sqrt{|x - y|^2 + h^2(f_2(x) - f_2(y))^2}} - \frac{1}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}} \right), \tag{2.5} \]
\[ I_h(\tilde{m}_h) = h \int_\omega \int_\omega (\tilde{m}_h \cdot e_3)(x, f_1)(\tilde{m}_h \cdot e_3)(y, f_1) \Gamma_h^1(x, y) \]
\[ + h \int_\omega \int_\omega (\tilde{m}_h \cdot e_3)(x, f_2)(\tilde{m}_h \cdot e_3)(y, f_2) \Gamma_h^2(x, y). \tag{2.6} \]

We will need the following propositions.

**Proposition 2.1.** Assume that \(f_1, f_2\) satisfy the hypothesis (H) and
\[ \int_\Omega \left( |\nabla' \tilde{m}_h|^2 + \frac{1}{h^2} \left( \frac{\partial \tilde{m}_h}{\partial z} \right)^2 \right) \leq C. \]
Then
\[
\frac{1}{h^2} \int_{\mathbb{R}^3} |\nabla u_h|^2 = \|\text{div}_p \tilde{m}_h + \tilde{m}_h (f_1) \cdot \nabla f_1 - \tilde{m}_h (f_2) \cdot \nabla f_2\|_{H^{-1/2}(\omega)}^2 + \frac{1}{h^2} f_h(\tilde{m}_h) + O(1)\|\tilde{m}_h \cdot e_3\|_{L^2(\omega)} + O(1)\|\tilde{m}_h \cdot e_3\|_{L^2(\omega)} + O(h),
\]
where \(\text{div}_p \tilde{m}_h(x) = \int_{f_2}^{f_1} \text{div}_p \tilde{m}_h(x, s) \, ds\) and \(\text{div}_p\) stands for a plane divergence.

**Proposition 2.2.** Assume that \(f_1, f_2\) satisfy the hypothesis \((H)\),
\[
\int_{\Omega} \left( |\nabla' \tilde{m}_h|^2 + \frac{1}{h^2} \left( \frac{\partial \tilde{m}_h}{\partial z} \right)^2 \right) + \frac{1}{h^2} I_h(\tilde{m}_h) \leq C \text{ and } \tilde{m}_h \to m \text{ weakly in } H^1(\Omega).
\]
Then

1. \(\liminf \frac{1}{h^2} I_h(\tilde{m}_h) \geq 0\),
2. \(\|\tilde{m}_h \cdot e_3\|_{L^2(\omega)} \to 0\) for \(i = 1, 2\).

Using Propositions 2.1 and 2.2, we will prove the following two theorems.

**Theorem 2.1.** Suppose \(\frac{d^2}{h^2} \to \infty\) as \(h \to 0\), \(f_1\) and \(f_2\) satisfy hypothesis \((H)\). Then we have:

1. if \(E_h(\tilde{m}_h) \leq C\) then \(\tilde{m}_h \to m\) strongly in \(H^1(\Omega; S^2)\) (may be for a subsequence), \(m = \text{const. and } m_3 = 0\);
2. \(E_h \to^\Gamma E_1\) in \(H^1(\Omega; S^2)\),

where
\[
E_1(m) = \left\{ \begin{array}{ll} \|m \cdot (\nabla f_1 - \nabla f_2)\|_{H^{-1/2}(\omega)}^2 & \text{if } m = \text{const. and } m_3 = 0 \quad (2.7) \\ +\infty & \text{otherwise}. \end{array} \right.
\]

**Theorem 2.2.** Suppose \(\frac{d^2}{h^2} \to \alpha\) as \(h \to 0\) and \(f_1\) and \(f_2\) satisfy hypothesis \((H)\), then we have:

1. if \(E_h(\tilde{m}_h) \leq C\) then \(\tilde{m}_h \to m\) weakly in \(H^1(\Omega; S^2)\) (may be for a subsequence), \(m = m(x)\) for \(x \in \omega\) and \(m_3 = 0\);
2. \(E_h \to^\Gamma E_2\) in \(H^1_w(\Omega; S^2)\),

where
\[
E_2(m) = \left\{ \begin{array}{ll} \alpha \int_{\omega} |\nabla m|^2(f_1 - f_2) + \|\text{div} (m(f_1 - f_2))\|_{H^{-1/2}(\omega)}^2 & \text{if } m = m(x), \ m_3 = 0 \quad (2.8) \\ +\infty & \text{otherwise}. \end{array} \right.
\]
3. Proof of $\Gamma$-Convergence

In this section we are going to prove the $\Gamma$-convergence stated as Theorems 2.1 and 2.2 in Sec. 2. Since the proofs are very similar, we will prove only Theorem 2.2.

The proof for Theorem 2.1 follows along the same lines.

**Proof of Theorem 2.2.** Let us prove the first statement. If $E_h(\tilde{m}_h) \leq C$ and $\frac{d}{h} \to 0$ then there exists a subsequence (not relabeled) such that $\tilde{m}_h \to m$ weakly in $H^1(\Omega)$. Moreover we may deduce that $m = m(x, y)$ is independent of $z$ variable. From Proposition 2.1 we obtain $\frac{1}{h^2}\|I_h(\tilde{m}_h)\| \leq C$ and then using Proposition 2.2 we have $\|(\tilde{m}_h \cdot e_3)(f_i)\|_{L^2(\omega)} \to 0$ for $i = 1, 2$. Since $m$ is independent of $z$, we see that $m_3 = 0$. Now let us prove $\Gamma$-convergence.

- *if $\tilde{m}_h \to m$ weakly in $H^1(\Omega)$ then* $\liminf E_h(\tilde{m}_h) \geq E(m)$.

First we consider the case $m_3 = 0$ and $m = m(x)$ for $x \in \omega$. Let us extract a subsequence $\{\tilde{m}_{h_k}\} \subset \{\tilde{m}_h\}$ such that $\liminf E_h(\tilde{m}_h) = \lim E_h(\tilde{m}_{h_k})$. We assume that $E_h(\tilde{m}_{h_k}) \leq C$ since otherwise

$$\liminf E_h(\tilde{m}_h) = \lim E_h(\tilde{m}_{h_k}) = \infty \geq E(m)$$

and there is nothing to prove.

By Proposition 2.1 we have $\frac{1}{h^3}\|I_h(\tilde{m}_{h_k})\| \leq C$ and hence using Proposition 2.2 we obtain $\liminf \frac{1}{h^3}\|\tilde{m}_{h_k}\| \geq 0$ and $\|(\tilde{m}_{h_k} \cdot e_3)(f_i)\|_{L^2(\omega)} \to 0$ for $i = 1, 2$. Applying Proposition 2.1 once again and noting that $\operatorname{div}_p m_{h_k} \to (f_1 - f_2)\operatorname{div}_p m$ weakly in $L^2(\omega)$ and $\tilde{m}_{h_k} \to m$ strongly in $L^2(\partial \Omega)$ for $i = 1, 2$ we deduce

$$\liminf \frac{1}{h^2} \int_{R^3} \|\nabla u_{h_k}\|^2 \geq \|(f_1 - f_2)\operatorname{div}_p m + m \cdot (\nabla f_1 - \nabla f_2)\|_{H^{-1/2}(\omega)}^2$$

$$= \|\operatorname{div}_p (m(f_1 - f_2))\|_{H^{-1/2}(\omega)}^2.$$

Now since $\tilde{m}_{h_k} \to m$ weakly in $H^1(\Omega)$ we have

$$\liminf \int_{\Omega} \|\nabla' \tilde{m}_{h_k}\|^2 \geq \int_{\omega} |\nabla' m|^2(f_1 - f_2).$$

Therefore we obtain

$$\liminf E_h(\tilde{m}_h) = \lim E_h(\tilde{m}_{h_k}) \geq \liminf \int_{\Omega} \|\nabla' \tilde{m}_{h_k}\|^2 + \liminf \frac{1}{h^2} \int_{R^3} |\nabla u_{h_k}|^2$$

$$\geq \alpha \int_{\omega} |\nabla' m|^2(f_1 - f_2) + \|\operatorname{div}_p (m(f_1 - f_2))\|_{H^{-1/2}(\omega)}^2 = E(m).$$

Now let us assume that one of the conditions $m = m(x, y)$ or $m_3 = 0$ is violated. If $m = m(x, y)$ is not satisfied, then we have

$$\liminf E_h(m_h) \geq \int_{\Omega} \frac{1}{h^2} \left(\frac{\partial m_h}{\partial z}\right)^2 \geq \infty = E(m).$$
If \( m_3 = 0 \) does not hold, then we may extract a subsequence \( \{ \tilde{m}_h \} \subset \{ \tilde{m}_h \} \) such that \( \liminf E_h(\tilde{m}_h) = \lim E_h(\tilde{m}_h) \) and assume that \( E_h(\tilde{m}_h) \leq C \). By Proposition 2.1 we may show that \( \frac{1}{\pi^2} |I_h(\tilde{m}_h)| \leq C \) and by Proposition 2.2 we obtain \( \| (\tilde{m}_h \cdot e_3)(f_i) \|_{L^2(\omega)} \to 0 \) for \( i = 1, 2 \). We also know that \( m \) is independent of \( z \) and this implies \( m_3 = 0 \), so we have a contradiction. Therefore \( \lim E_h(\tilde{m}_h) = \infty \) and

\[
\liminf E_h(\tilde{m}_h) \geq \infty = E(m).
\]

- for every \( m \in H^1(\Omega; S) \) there exists \( \tilde{m}_h \to m \) weakly in \( H^1(\Omega) \) such that \( \lim E_h(\tilde{m}_h) = E(m) \).

Let us assume first that \( m_3 = 0 \) and \( m = m(x) \) for \( x \in \omega \) then we may construct a sequence by taking \( \tilde{m}_h = m \). Plugging this sequence in the expression for energy (2.2) we have

\[
E_h(m) = \frac{d^2}{h} \int_{\Omega} |\nabla m|^2 + \frac{1}{h^2} \int_{\mathbb{R}^3} |\nabla u|^2.
\]

By Proposition 2.1 we know that in this case

\[
\frac{1}{h^2} \int_{\mathbb{R}^3} |\nabla u|^2 = \| \text{div}_p (m(f_1 - f_2)) \|_{H^{-1/2}(\omega)}^2 + O(h).
\]

Taking a limit as \( h \to 0 \) we obtain

\[
\lim E_h(m) = \alpha \int_{\omega} |\nabla m|^2 (f_1 - f_2) + \| \text{div}_p (m(f_1 - f_2)) \|_{H^{-1/2}(\omega)}^2 = E(m).
\]

If at least one of the conditions \( m = m(x) \) or \( m_3 = 0 \) is violated, then we already know that

\[
\liminf E_h(m_h) \geq \infty,
\]

hence for any such sequence we have

\[
\lim E_h(m_h) = \infty = E(m).
\]

The theorem is proved.

4. Asymptotics of Magnetostatic Energy

In this section we are going to simplify the nonlocal term as \( h \to 0 \) and prove Propositions 2.1 and 2.2. We essentially follow the ideas of Ref. 7, and make repeated use of the following result:

**Lemma 4.1.** Assume \( f, g \in L^2(\omega) \) and \( 0 \leq a \leq b \) are fixed numbers, then

\[
\int_{\omega} \int_{\omega} f(x)g(y) \left( \frac{1}{\sqrt{|x - y|^2 + a^2}} - \frac{1}{\sqrt{|x - y|^2 + b^2}} \right) dx \, dy \leq C b \| f \|_{L^2(\omega)} \| g \|_{L^2(\omega)}.
\]
Proof of Proposition 2.1. The magnetostatic energy may be written as (see (2.2))

\[ h^2 E_{\text{magn}}(m_h) = \int_{\mathbb{R}^3} |\nabla u_h|^2 = -\int_{\Omega_h} \nabla u_h \cdot m_h \]
\[ = \int_{\Omega_h} u_h \text{div} m_h - \int_{\partial \Omega_h} u_h (m_h \cdot n). \]  

(4.1)

Solving the equation for \( u_h \) we also have

\[ 4\pi u_h(x) = \int_{\Omega_h} \frac{1}{|x - y|} \text{div} m_h(y) - \int_{\partial \Omega_h} \frac{1}{|x - y|} (m_h \cdot n)(y). \]  

(4.2)

Plugging the expression (4.2) for \( u_h \) into formula (4.1) for magnetostatic energy, we obtain

\[ 4\pi h^2 E_{\text{magn}}(m_h) = \int_{\Omega_h} \int_{\Omega_h} \frac{1}{|x - y|} \text{div} m_h(y) \text{div} m_h(x) \]
\[ + \int_{\partial \Omega_h} \int_{\partial \Omega_h} \frac{1}{|x - y|} (m_h \cdot n)(y)(m_h \cdot n)(x) \]
\[ - 2 \int_{\partial \Omega_h} \int_{\partial \Omega_h} \frac{1}{|x - y|} \text{div} m_h(y)(m_h \cdot n)(x). \]  

(4.3)

Below we are going to describe three terms in (4.3) as “bulk–bulk term”, “boundary–boundary term”, and “bulk–boundary term”. We will expand these terms and then combine them in a suitable manner.

Bulk-bulk term.
Let us expand the bulk–bulk term:

\[ \int_{\Omega_h} \int_{\Omega_h} \frac{1}{|x - y|} \text{div} m_h(y) \text{div} m_h(x) = A_1 + A_2 + A_3, \]  

(4.4)

where we use the following notation

\[ A_1 = \int_{\omega} \int_{\omega} \int_{h_{f_1}}^{h_{f_2}} \int_{h_{f_1}}^{h_{f_2}} \text{div}_p m_h(x, s, t) \text{div}_p m_h(y, t) \frac{1}{\sqrt{|x - y|^2 + (s - t)^2}}, \]
\[ A_2 = \int_{\omega} \int_{\omega} \int_{h_{f_1}}^{h_{f_2}} \int_{h_{f_1}}^{h_{f_2}} \frac{\partial (m_{h,e_3})}{\partial z}(x, s) \frac{\partial (m_{h,e_3})}{\partial z}(y, t) \frac{1}{\sqrt{|x - y|^2 + (s - t)^2}}, \]  

(4.5)

\[ A_3 = 2 \int_{\omega} \int_{\omega} \int_{h_{f_1}}^{h_{f_2}} \int_{h_{f_1}}^{h_{f_2}} \text{div}_p m_h(x, s) \frac{\partial (m_{h,e_3})}{\partial z}(y, t) \frac{1}{\sqrt{|x - y|^2 + (s - t)^2}}. \]

Here \( \text{div}_p m_h \) denotes a plane divergence.
Let us expand bulk–boundary term:

\[ 2 \int_{\partial \Omega} \int_{\Omega_h} \frac{1}{|x - y|} \text{div} m_h(y)(m_h \cdot n)(x) = B_1 + B_2 + B_3 + B_4, \]  

where

\[ B_1 = 2 \int_{S_1} \int_{\Omega_h} \frac{\text{div}_p m_h(x)(m_h \cdot n)(y)}{|x - y|}, \]

\[ B_2 = 2 \int_{S_1} \int_{\Omega_h} \frac{\partial (m_h \cdot e_3)}{\partial n}(x)(m_h \cdot n)(y)}{|x - y|}, \]

\[ B_3 = 2 \int_{S_2} \int_{\Omega_h} \frac{\text{div}_p m_h(x)(m_h \cdot n)(y)}{|x - y|}, \]

\[ B_4 = 2 \int_{S_2} \int_{\Omega_h} \frac{\partial (m_h \cdot e_3)}{\partial n}(x)(m_h \cdot n)(y)}{|x - y|}. \]

Now let us elaborate on it:

\[ \int_{S_1} f(x) = \int_{\omega} f(x, h f_1) \sqrt{1 + h^2 |\nabla f_1|^2}, \]

\[ \int_{S_2} f(x) = \int_{\omega} f(x, h f_2) \sqrt{1 + h^2 |\nabla f_2|^2}. \]

Normal vectors to \(S_1\) and \(S_2\) are

\[ n_1 = \frac{1}{\sqrt{1 + h^2 |\nabla f_1|^2}} (-h \nabla f_1, 1), \]

\[ n_2 = \frac{1}{\sqrt{1 + h^2 |\nabla f_2|^2}} (h \nabla f_2, -1). \]

Therefore we have

\[ B_1 = 2 \int_{\omega} \int_{h f_1} \text{div}_p m_h(x, s)((m_h \cdot e_3) - h(m_h \cdot \nabla f_1))(y, h f_1) \sqrt{|x - y|^2 + (s - h f_1)^2}, \]

\[ B_2 = 2 \int_{\omega} \int_{h f_1} \frac{\partial (m_h \cdot e_3)}{\partial n}(x, s)((m_h \cdot e_3) - h(m_h \cdot \nabla f_1))(y, h f_1) \sqrt{|x - y|^2 + (s - h f_1)^2}, \]

\[ B_3 = 2 \int_{\omega} \int_{h f_2} \text{div}_p m_h(x, s)(-(m_h \cdot e_3) + h(m_h \cdot \nabla f_2))(y, h f_2) \sqrt{|x - y|^2 + (s - h f_2)^2}, \]

\[ B_4 = 2 \int_{\omega} \int_{h f_2} \frac{\partial (m_h \cdot e_3)}{\partial n}(x, s)(-(m_h \cdot e_3) + h(m_h \cdot \nabla f_2))(y, h f_2) \sqrt{|x - y|^2 + (s - h f_2)^2}. \]
Now let us combine the terms in the following fashion

$$B_1 + B_3 = D_1 + D_2,$$

where

$$D_1 = 2 \int_\omega \int_\omega \int_{h f_2}^{h f_1} \text{div}_p m_h(x, s) \left( \frac{(m_h \cdot e_3)(y, h f_1)}{\sqrt{|x - y|^2 + (s - h f_1)^2}} \right)$$

$$- \frac{(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + (s - h f_2)^2}} \right),$$

$$D_2 = 2h \int_\omega \int_\omega \int_{h f_2}^{h f_1} \text{div}_p m_h(x, s) \left( \frac{(m_h \cdot \nabla f_2)(y, h f_1)}{\sqrt{|x - y|^2 + (s - h f_1)^2}} \right)$$

$$- \frac{(m_h \cdot \nabla f_1)(y, h f_2)}{\sqrt{|x - y|^2 + (s - h f_2)^2}} \right).$$

In the same way we obtain

$$B_2 + B_4 = D_3 + D_4,$$

where

$$D_3 = 2 \int_\omega \int_\omega \int_{h f_2}^{h f_1} \frac{\partial (m_h \cdot e_3)}{\partial z} (x, s) \left( \frac{(m_h \cdot e_3)(y, h f_1)}{\sqrt{|x - y|^2 + (s - h f_1)^2}} \right)$$

$$- \frac{(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + (s - h f_2)^2}} \right),$$

$$D_4 = 2h \int_\omega \int_\omega \int_{h f_2}^{h f_1} \frac{\partial (m_h \cdot e_3)}{\partial z} (x, s) \left( \frac{(m_h \cdot \nabla f_2)(y, h f_1)}{\sqrt{|x - y|^2 + (s - h f_1)^2}} \right)$$

$$- \frac{(m_h \cdot \nabla f_1)(y, h f_2)}{\sqrt{|x - y|^2 + (s - h f_2)^2}} \right).$$

Now we have

$$D_1 = E_1 + F_1,$$

where

$$E_1 = 2 \int_\omega \int_\omega \int_{h f_2}^{h f_1} \text{div}_p m_h(x, s) \left( (m_h \cdot e_3)(y, h f_1) - (m_h \cdot e_3)(y, h f_2) \right)$$

$$\left( \frac{(m_h \cdot e_3)(y, h f_1)}{\sqrt{|x - y|^2 + (s - h f_1)^2}} \right),$$

$$F_1 = 2h \int_\omega \int_\omega \int_{h f_2}^{h f_1} \text{div}_p m_h(x, s) \left( \frac{(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + (s - h f_2)^2}} \right)$$

$$- \frac{(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + (s - h f_2)^2}} \right).$$
and hence
\[ E_1 = 2 \int_\omega \int_\omega \int_{h f_1}^{h f_1} \int_{h f_2}^{h f_2} \text{div}_p m_h(x, s) \frac{\partial (m_h, e_3)}{\partial z} (y, t) \frac{\sqrt{|x-y|^2 + (s-h f_2)^2}}{|x-y|^2 + (s-h f_2)^2} \, ds. \]

Here we have used the following identity:
\[ \int_{h f_2}^{h f_1} \frac{\partial (m_h, e_3)}{\partial z} (x, s) ds = (m_h, e_3)(x, h f_1) - (m_h, e_3)(x, h f_2). \]

By the same arguments we may show that
\[ D_2 = E_2 + F_2, \quad D_3 = E_3 + F_3, \quad D_4 = E_4 + F_4, \]

where
\begin{align*}
E_2 & = 2h \int_\omega \int_\omega \int_{h f_2}^{h f_1} \text{div}_p m_h(x, s) \left( \frac{(m_h \cdot \nabla f_2)(y, h f_2) - (m_h \cdot \nabla f_1)(y, h f_1)}{\sqrt{|x-y|^2 + (s-h f_2)^2}} \right), \\
E_3 & = 2h \int_\omega \int_\omega \int_{h f_2}^{h f_1} \frac{\partial (m_h, e_3)}{\partial z} (x, s) \frac{\partial (m_h, e_3)}{\partial z} (y, t) \frac{\sqrt{|x-y|^2 + (s-h f_1)^2}}{|x-y|^2 + (s-h f_1)^2}, \\
E_4 & = 2h \int_\omega \int_\omega \int_{h f_2}^{h f_1} \frac{\partial (m_h, e_3)}{\partial z} (x, s) \left[ \left( (m_h \cdot \nabla f_2)(y, h f_2) - (m_h \cdot \nabla f_1)(y, h f_1) \right) \right], \\
F_2 & = 2h \int_\omega \int_\omega \int_{h f_2}^{h f_1} \text{div}_p m_h(x, s) \left( \frac{(m_h \cdot \nabla f_1)(y, h f_1)}{\sqrt{|x-y|^2 + (s-h f_2)^2}} \right), \\
F_3 & = 2h \int_\omega \int_\omega \int_{h f_2}^{h f_1} \frac{\partial (m_h, e_3)}{\partial z} (x, s) \left( \frac{(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x-y|^2 + (s-h f_1)^2}} \right), \\
F_4 & = 2h \int_\omega \int_\omega \int_{h f_2}^{h f_1} \frac{\partial (m_h, e_3)}{\partial z} (x, s) \left( \frac{(m_h \cdot \nabla f_1)(y, h f_1)}{\sqrt{|x-y|^2 + (s-h f_2)^2}} \right).
\end{align*}

Let us combine some of the terms and use Lemma 4.1 to estimate them
\begin{align*}
|A_3 - E_1| & \leq C h^3 \left( \frac{1}{h^2} \left\| \frac{\partial \tilde{m}_h}{\partial z} \right\|_{L^2(\Omega)}^2 + \| \text{div}_p \tilde{m}_h \|_{L^2(\Omega)}^2 \right), \\
|F_1| + |F_3| & \leq C h^2 \|(m_h \cdot e_3)(f_2)\|_{L^2(\omega)} \left( \frac{1}{h} \left\| \frac{\partial \tilde{m}_h}{\partial z} \right\|_{L^2(\Omega)} + \| \text{div}_p \tilde{m}_h \|_{L^2(\Omega)} \right),
\end{align*}
Here we use the following notation

\[ |F_1| \leq Ch^3 \left( \frac{1}{h^2} \left\| \frac{\partial \tilde{m}}{\partial z} \right\|_{L^2(\Omega)}^2 + \left\| \tilde{m} \cdot \nabla f_1 \right\|_{L^2(\omega)}^2 \right), \]

\[ |F_2| \leq Ch^3 \left( \left\| \text{div}_p \tilde{m} \right\|_{L^2(\Omega)}^2 + \left\| \tilde{m} \cdot \nabla f_1 \right\|_{L^2(\omega)}^2 \right). \]

So we may write

\[ 4\pi h^2 \mathcal{E}_{\text{magn}}(m_h) = A_1 + A_2 - E_2 - E_3 - E_4 + \int_{\partial \Omega_h} \int_{\partial \Omega_h} \frac{1}{|x-y|} (m_h \cdot n)(y)(m_h \cdot n)(x) \]

\[ + O(h^3) \left( \frac{1}{h^2} \left\| \frac{\partial \tilde{m}}{\partial z} \right\|_{L^2(\Omega)}^2 + \left\| \text{div}_p \tilde{m} \right\|_{L^2(\omega)}^2 + \left\| \tilde{m} \cdot \nabla f_1 \right\|_{L^2(\omega)}^2 \right) \]

\[ + O(h^3) \left( \left\| \tilde{m} \cdot e_3 \right\|_{L^2(\omega)}(f_2) \left( \frac{1}{h} \left\| \frac{\partial \tilde{m}}{\partial z} \right\|_{L^2(\Omega)} + \left\| \text{div}_p \tilde{m} \right\|_{L^2(\Omega)} \right) \right). \quad (4.16) \]

**Boundary–boundary term.**

Similarly, we expand and reorganize the boundary–boundary term.

\[ \int_{\partial \Omega_h} \int_{\partial \Omega_h} \frac{1}{|x-y|} (m_h \cdot n)(y)(m_h \cdot n)(x) = C_1 + C_2 + C_3. \quad (4.17) \]

Here we use the following notation

\[ C_1 = \int_{S_1} \int_{S_1} \frac{(m_h \cdot n_1)(x)(m_h \cdot n_1)(y)}{|x-y|}, \]

\[ C_2 = \int_{S_2} \int_{S_2} \frac{(m_h \cdot n_2)(x)(m_h \cdot n_2)(y)}{|x-y|}, \]

\[ C_3 = 2 \int_{S_1} \int_{S_2} \frac{(m_h \cdot n_2)(x)(m_h \cdot n_1)(y)}{|x-y|}. \quad (4.18) \]

Using change of variables we obtain

\[ C_1 = \int_{\omega} \int_{\omega} \frac{((m_h \cdot e_3) - h(m_h \cdot \nabla f_1))(x, h f_1)((m_h \cdot e_3) - h(m_h \cdot \nabla f_1))(y, h f_1)}{\sqrt{|x-y|^2 + h^2(f_1(x) - f_1(y))^2}}, \]

\[ C_2 = \int_{\omega} \int_{\omega} \frac{((m_h \cdot e_3) - h(m_h \cdot \nabla f_2))(x, h f_2)((m_h \cdot e_3) - h(m_h \cdot \nabla f_2))(y, h f_2)}{\sqrt{|x-y|^2 + h^2(f_2(x) - f_2(y))^2}}, \]

\[ C_3 = 2 \int_{\omega} \int_{\omega} \frac{((m_h \cdot e_3) - h(m_h \cdot \nabla f_1))(x, h f_1)((m_h \cdot e_3) + h(m_h \cdot \nabla f_2))(y, h f_2)}{\sqrt{|x-y|^2 + h^2(f_1(x) - f_2(y))^2}}. \quad (4.19) \]

Now we have

\[ C_1 = G_1 + H_1 + I_1, \quad C_2 = G_2 + H_2 + I_2, \quad C_3 = G_3 + H_3 + I_3 + I_4, \]
where

\[ G_1 = \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(y, h f_1)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_1(y))^2}}, \]

\[ G_2 = \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_2)(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_2(x) - f_2(y))^2}}, \]

\[ G_3 = -2 \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}}, \]

\[ H_1 = h^2 \int_\omega \int_\omega \frac{(m_h \cdot \nabla f_1)(x, h f_1)(m_h \cdot \nabla f_1)(y, h f_1)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_1(y))^2}}, \]

\[ H_2 = h^2 \int_\omega \int_\omega \frac{(m_h \cdot \nabla f_2)(x, h f_2)(m_h \cdot \nabla f_2)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_2(x) - f_2(y))^2}}, \]

\[ H_3 = -2h^2 \int_\omega \int_\omega \frac{(m_h \cdot \nabla f_1)(x, h f_1)(m_h \cdot \nabla f_2)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}}, \]

\[ I_1 = -2h \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_1)(m_h \cdot \nabla f_1)(y, h f_1)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_1(y))^2}}, \]

\[ I_2 = -2h \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_2)(m_h \cdot \nabla f_2)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_2(x) - f_2(y))^2}}, \]

\[ I_3 = 2h \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_1)(m_h \cdot \nabla f_2)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}}, \]

\[ I_4 = 2h \int_\omega \int_\omega \frac{(m_h \cdot \nabla f_1)(x, h f_1)(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}}. \]

Notice that

\[ G_1 + G_2 + G_3 \]

\[ = \int_\omega \int_\omega \left( \frac{(m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(y, h f_1)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_1(y))^2}} - \frac{(m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(y, h f_1)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}} \right) \]

\[ + \int_\omega \int_\omega \left( \frac{(m_h \cdot e_3)(x, h f_2)(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_2(x) - f_2(y))^2}} - \frac{(m_h \cdot e_3)(x, h f_2)(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}} \right) \]

\[ + \int_\omega \int_\omega \left( \frac{(m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(y, h f_1)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_1(y))^2}} + \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_2)(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}} \right) \]

\[ - 2 \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}}. \]
and rearranging terms we obtain

\[
G_1 + G_2 + G_3 = h \int_\omega \int_\omega (m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(x, h f_1) \Gamma_h^1(x, y) + h \int_\omega \int_\omega (m_h \cdot e_3)(x, h f_2)(m_h \cdot e_3)(x, h f_2) \Gamma_h^2(x, y) + \int_\Omega_h \int_{\partial \Omega_h} \frac{\partial (m_h \cdot e_3)(x, s)}{\partial x} \frac{\partial (m_h \cdot e_3)(y, t)}{\partial y} \sqrt{|x - y|^2 + h^2 (f_1(x) - f_2(y))^2} + \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_2)(m_h \cdot e_3)(y, h f_1) - (m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + h^2 (f_1(x) - f_2(y))^2}}.
\]

(4.22)

We define

\[
K_1 = \int_\Omega_h \int_\partial \Omega_h \frac{\partial (m_h \cdot e_3)(x, s)}{\partial x} \frac{\partial (m_h \cdot e_3)(y, t)}{\partial y} \sqrt{|x - y|^2 + h^2 (f_1(x) - f_2(y))^2}.
\]

After rearrangement we also have the following estimate

\[
\int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_2)(m_h \cdot e_3)(y, h f_1) - (m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(y, h f_2)}{\sqrt{|x - y|^2 + h^2 (f_1(x) - f_2(y))^2}} = \int_\omega \int_\omega \int_{h f_2}^{h f_1} \frac{\partial (m_h \cdot e_3)(x, s)}{\partial x} (x, s) \left( \frac{(m_h \cdot e_3)(y, h f_1)}{\sqrt{|x - y|^2 + h^2 (f_1(x) - f_2(y))^2}} - \frac{(m_h \cdot e_3)(y, h f_1)}{\sqrt{|x - y|^2 + h^2 (f_1(x) - f_2(y))^2}} \right)
\]

\[
\leq C h^2 \frac{1}{h} \left\| \frac{\partial m_h}{\partial x} \right\|_{L^2(\Omega)} \left\| (\tilde{m}_h \cdot e_3)(f_1) \right\|_{L^2(\omega)}.
\]

(4.23)

So we have

\[
G_1 + G_2 + G_3 = h \int_\omega \int_\omega (m_h \cdot e_3)(x, h f_1)(m_h \cdot e_3)(y, h f_1) \Gamma_h^1(x, y) + h \int_\omega \int_\omega (m_h \cdot e_3)(x, h f_2)(m_h \cdot e_3)(y, h f_2) \Gamma_h^2(x, y) + K_1 + O(h^2) \frac{1}{h} \left\| \frac{\partial m_h}{\partial x} \right\|_{L^2(\Omega)} \left\| (\tilde{m}_h \cdot e_3)(f_1) \right\|_{L^2(\omega)}.
\]

(4.24)

Now we calculate

\[
I_1 + I_2 + I_3 + I_4 = 2h \int_\omega \int_\omega \frac{(m_h \cdot e_3)(x, h f_1)((m_h \cdot \nabla f_2)(y, h f_2) - (m_h \cdot \nabla f_2)(y, h f_1))}{\sqrt{|x - y|^2 + h^2 (f_1(x) - f_2(y))^2}} + 2h \int_\omega \int_\omega \frac{(m_h \cdot e_3)(y, h f_2)((m_h \cdot \nabla f_1)(x, h f_1) - (m_h \cdot \nabla f_2)(x, h f_2))}{\sqrt{|x - y|^2 + h^2 (f_1(x) - f_2(y))^2}}.
\]
and using Lemma 4.1 again we obtain

\[
+ 2h \int_\omega \int_{\Omega_h} (m_h \cdot e_3)(x, h f_1) \left( \frac{(m_h \cdot \nabla f_1)(y, h f_1)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}} \right)
- \frac{(m_h \cdot \nabla f_1)(y, h f_1)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}}
\]

\[
+ 2h \int_\omega \int_{\Omega_h} (m_h \cdot e_3)(y, h f_2) \left( \frac{(m_h \cdot \nabla f_2)(x, h f_2)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}} \right)
- \frac{(m_h \cdot \nabla f_2)(x, h f_2)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}} \right).
\]

(4.25)

It follows from Lemma 4.1 and some rearrangement that

\[ I_1 + I_2 + I_3 + I_4 = 2h \int_\omega \int_{\Omega_h} \frac{\partial (m_h \cdot e_3)}{\partial x}(x, s) \left( (m_h \cdot \nabla f_2)(y, h f_2) - (m_h \cdot \nabla f_1)(y, h f_1) \right) \]

\[ + 2h \int_\omega \int_{\Omega_h} (m_h \cdot e_3)(x, h f_2) \left( \frac{(m_h \cdot \nabla f_2)(x, h f_2) - (m_h \cdot \nabla f_1)(y, h f_1)}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(y))^2}} \right)
- \frac{(m_h \cdot \nabla f_2)(x, h f_1) - (m_h \cdot \nabla f_1)(y, h f_2)}{\sqrt{|x - y|^2 + h^2(f_2(y) - f_1(x))^2}} \right)
\]

\[ + O(h^2) \left( \|(\tilde{m}_h \cdot e_3)(f_2)\|_{L^2_\omega} + \|(\tilde{m}_h \cdot e_3)(f_1)\|_{L^2_\omega} \right)
\cdot \left( \|\tilde{m}_h \cdot \nabla f_1\|_{L^2_\omega} + \|\tilde{m}_h \cdot \nabla f_2\|_{L^2_\omega} \right). \]

(4.26)

and using Lemma 4.1 again we obtain

\[ I_1 + I_2 + I_3 + I_4 = 2h \int_\omega \int_{\Omega_h} \frac{\partial (m_h \cdot e_3)}{\partial x}(x, s) \left( (m_h \cdot \nabla f_2)(y, h f_2) - (m_h \cdot \nabla f_1)(y, h f_1) \right) \]

\[ + O(h^2) \left( \|(\tilde{m}_h \cdot e_3)(f_2)\|_{L^2_\omega} + \|(\tilde{m}_h \cdot e_3)(f_1)\|_{L^2_\omega} \right)
\cdot \left( \|\tilde{m}_h \cdot \nabla f_1\|_{L^2_\omega} + \|\tilde{m}_h \cdot \nabla f_2\|_{L^2_\omega} \right). \]

(4.27)

We define

\[ K_2 = 2h \int_\omega \int_{\Omega_h} \frac{\partial (m_h \cdot e_3)}{\partial x}(x, s) \left( (m_h \cdot \nabla f_2)(y, h f_2) - (m_h \cdot \nabla f_1)(y, h f_1) \right) \]
Combining terms and using Lemma 4.1 we have

\[ |A_2 - E_3 + K_1| \leq Ch^3 \left( \frac{1}{h^2} \left\| \frac{\partial \tilde{m}_h}{\partial z} \right\|_{L^2(\Omega)}^2 \right), \]

\[ |K_2 - E_4| \leq Ch^3 \left( \frac{1}{h^2} \left\| \frac{\partial \tilde{m}_h}{\partial z} \right\|_{L^2(\Omega)}^2 + \| \tilde{m}_h \cdot \nabla f_1 \|_{L^2(\omega)}^2 + \| \tilde{m}_h \cdot \nabla f_2 \|_{L^2(\omega)}^2 \right). \]

So we have

\[ 4\pi h^2 E_{\text{magn}}(m_h) = A_1 - E_2 + H_1 + H_2 + H_3 + I_h(\tilde{m}_h) \]

\[ + O(h^3) \left( \left\| \frac{\partial \tilde{m}_h}{\partial z} \right\|_{L^2(\omega)}^2 + \| \text{div}_p \tilde{m}_h \|_{L^2(\omega)}^2 \right) \]

\[ + \| \tilde{m}_h \cdot \nabla f_1 \|_{L^2(\omega)}^2 + \| \tilde{m}_h \cdot \nabla f_2 \|_{L^2(\omega)}^2 \]

\[ + O(h^2) \left( \| (\tilde{m}_h \cdot e_3)(f_2) \|_{L^2(\omega)} + \| (\tilde{m}_h \cdot e_3)(f_1) \|_{L^2(\omega)} \right) \]

\[ \cdot \left( \frac{1}{h} \left\| \frac{\partial \tilde{m}_h}{\partial z} \right\|_{L^2(\omega)} + \| \text{div}_p \tilde{m}_h \|_{L^2(\omega)} \right) \]

\[ + \| \tilde{m}_h \cdot \nabla f_1 \|_{L^2(\omega)} + \| \tilde{m}_h \cdot \nabla f_2 \|_{L^2(\omega)} \right). \]

Using assumptions of the proposition we have

\[ 4\pi h^2 E_{\text{magn}}(m_h) = A_1 - E_2 + H_1 + H_2 + H_3 + I_h(\tilde{m}_h) + O(h^3) \]

\[ + O(h^2) \left( \| (\tilde{m}_h \cdot e_3)(f_2) \|_{L^2(\omega)} + \| (\tilde{m}_h \cdot e_3)(f_1) \|_{L^2(\omega)} \right). \]  \tag{4.28}

Now let us simplify term \( A_1 \):

\[ A_1 = \int_\omega \int_\omega \int_{h f_2}^{h f_1} \int_{h f_2}^{h f_1} \frac{\text{div}_p m_h(x, s) \text{div}_p m_h(y, t)}{|x - y|} + R(h), \]

where

\[ R(h) = \int_\omega \int_\omega \int_{h f_2}^{h f_1} \int_{h f_2}^{h f_1} \left( \frac{\text{div}_p m_h(x, s) \text{div}_p m_h(y, t)}{\sqrt{|x - y|^2 + (s - t)^2}} - \frac{\text{div}_p m_h(x, s) \text{div}_p m_h(y, t)}{|x - y|} \right). \]

Estimating \( R(h) \leq Ch^3 \| \text{div}_p \tilde{m}_h \|_{L^2(\Omega)}^2 \) we have

\[ A_1 = h^2 \int_\omega \int_\omega \frac{\text{div}_p \tilde{m}_h(x) \text{div}_p \tilde{m}_h(y)}{|x - y|} + O(h^3). \]
By the same arguments we obtain

\[
E_2 = 2h^2 \int_\omega \frac{\text{div} \cdot \tilde{m}_h(x)((\tilde{m}_h \cdot \nabla f_2)(y, f_2) - (\tilde{m}_h \cdot \nabla f_1)(y, f_1))}{|x - y|} y + O(h^3),
\]

\[
H_1 = h^2 \int_\omega \frac{\tilde{m}_h \cdot \nabla f_1(x, f_1)(\tilde{m}_h \cdot \nabla f_1)(y, f_1)}{|x - y|} + O(h^3),
\]

\[
H_2 = h^2 \int_\omega \frac{\tilde{m}_h \cdot \nabla f_2(x, f_2)(\tilde{m}_h \cdot \nabla f_2)(y, f_2)}{|x - y|} + O(h^3),
\]

\[
H_3 = -2h^2 \int_\omega \frac{\tilde{m}_h \cdot \nabla f_1(x, f_1)(\tilde{m}_h \cdot \nabla f_2)(y, f_2)}{|x - y|} + O(h^3).
\]

Combining all the terms we obtain the result. The proposition is proved.

**Lemma 4.2.** We define

\[
G_h(x, y) = \frac{1}{h} \left( \frac{1}{|x - y|} - \frac{1}{\sqrt{|x - y|^2 + h^2(f_1(x) - f_2(x))^2}} \right),
\]

\[
g_h(x) = \int_{\mathbb{R}^2} a_h(y)G_h(x, y).
\]

If \(|a_h| \leq C, a_h \rightarrow a\) in \(L^2(\mathbb{R}^2)\) then \(g_h \rightarrow 2\pi a(f_1 - f_2)\) in \(L^2(\mathbb{R}^2)\).

**Proof.** First we notice that

\[
\int_{\mathbb{R}^2} G_h(x, y)dy = 2\pi(f_1(x) - f_2(x)),
\]

\[
0 \leq G_h(x, y) \leq K_h(x - y) = \frac{1}{h} \left( \frac{1}{|x - y|} - \frac{1}{\sqrt{|x - y|^2 + C^2 h^2}} \right),
\]

\[
\int_{\mathbb{R}^2} K_h(x) \leq C \quad \text{and for any fixed } \delta > 0, \quad \int_{|x| > \delta} K_h(x) \rightarrow 0.
\]

Now we have

\[
|g_h(x) - 2\pi a_h(x)(f_1(x) - f_2(x))| = \left| \int_{\mathbb{R}^2} (a_h(y)G_h(x, y) - a_h(x)G_h(x, y))dy \right|
\]

(making a change of variables \(\xi = x - y\))

\[
\leq \int_{\mathbb{R}^2} |a_h(\xi + x) - a_h(x)|K_h(\xi)d\xi \leq C \left( \int_{\mathbb{R}^2} |a_h(\xi + x) - a_h(x)|^2 K_h(\xi)d\xi \right)^{1/2}.
\]

Hence

\[
\int_{\mathbb{R}^2} |g_h(x) - 2\pi a_h(x)(f_1 - f_2)|^2dx \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |a_h(\xi + x) - a_h(x)|^2 K_h(\xi)d\xi dx.
\]
Interchanging the order of integration we obtain
\[
\int_{\mathbb{R}^2} |g_h(x) - 2\pi a_h(x)(f_1 - f_2)|^2 dx \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |a_h(\xi + x) - a_h(x)|^2 dx K_h(\xi) d\xi.
\]

Notice that
\[
\|a_h(\xi + x) - a_h(x)||_{L^2(\mathbb{R}^2)} \leq 2\|a_h - a||_{L^2(\mathbb{R}^2)} + \|a(\xi + x) - a(x)||_{L^2(\mathbb{R}^2)}.
\]

Using this inequality we have
\[
\int_{\mathbb{R}^2} |g_h(x) - 2\pi a_h(x)(f_1 - f_2)|^2 dx \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |a(\xi + x) - a(x)|^2 dx K_h(\xi) d\xi + o(h).
\]

Since \(\|a(\xi + x) - a(x)||_{L^2(\mathbb{R}^2)} \to 0\) as \(\xi \to 0\) we have
\[
\int_{\mathbb{R}^2} |g_h(x) - 2\pi a_h(x)(f_1 - f_2)|^2 dx \leq \varepsilon \int_{|\xi| \leq \delta} K_h(\xi) d\xi + C \int_{|\xi| > \delta} K_h(\xi) d\xi + o(h).
\]

Taking a limit as \(h \to 0\) we obtain
\[
\lim \int_{\mathbb{R}^2} |g_h(x) - 2\pi a_h(x)(f_1 - f_2)|^2 dx \leq C \varepsilon.
\]

Since it is true for any \(\varepsilon > 0\) and since \(a_h \to a\) in \(L^2(\mathbb{R}^2)\) we obtain the result. The lemma is proved.

**Proof of Proposition 2.2.** Let us prove that \(\|\langle \hat{m}_h \cdot e_3 \rangle(f_i)\|_{L^2(\omega)} \to 0\) for \(i = 1, 2\).

We notice that for any \(a, b \in L^\infty(\omega)\) and \(\beta > 0\)
\[
\int_{\omega} \int_{\omega} |a(x)||b(y)||\Gamma^i_h(x, y) - G_h(x, y)| \leq C \beta^i \int_{\omega} \int_{\omega} \frac{1}{|x - y|^2 \gamma + 2\beta} \leq C \beta^2
\]
for \(2\beta < \gamma\). Since \(|I_h(\hat{m}_h)| \leq C h^2\) this implies that
\[
|J_h(\hat{m}_h)| = \left| \frac{1}{2\pi} \int_{\omega} \int_{\omega} (\hat{m}_h \cdot e_3)(x, f_1)(\hat{m}_h \cdot e_3)(y, f_1) G_h(x, y) \right| \leq o(h).
\]

From Lemma 4.2 we deduce that
\[
\frac{1}{2\pi} J_h(\hat{m}_h) \rightarrow \int_{\omega} |(m \cdot e_3)(f_1)(x)|^2 (f_1(x) - f_2(x)) dx
+ \int_{\omega} |(m \cdot e_3)(f_2)(x)|^2 (f_1(x) - f_2(x)) dx \quad (4.30)
\]

Since \(|J_h(\hat{m}_h)| \leq o(h)\) we have \((m \cdot e_3)(f_i) = 0\) and therefore
\[
\lim \|\langle \hat{m}_h \cdot e_3 \rangle(f_i)\|_{L^2(\omega)} = \|\langle m \cdot e_3 \rangle(f_i)\|_{L^2(\omega)} = 0.
\]

Let us now prove that \(\liminf \frac{1}{h^3} J_h(\hat{m}_h) \geq 0\). We consider the following problem
\[
-\Delta v_h = \text{div} (m_{3,h} \chi_{\Omega_h}) \quad \text{in } \mathbb{R}^3,
\]
where \( m_{3,h} = (0, 0, (m_h \cdot e_3)) \). We find as before

\[
0 \leq \int_{\mathbb{R}^3} |\nabla v_h|^2 = A_2 - E_3 - F_3 + G_1 + G_2 + G_3 = A_2 - E_3 + K_1 - F_3 + I_h(\tilde{m}_h) + O(h^2) \frac{1}{h} \left\| \frac{\partial \tilde{m}_h}{\partial z} \right\|_{L^2(\Omega)} \| (\tilde{m}_h \cdot e_3)(f_2) \|_{L^2(\omega)} \\
= I_h(\tilde{m}_h) + O(h^3) + O(h^2) \left( \| (\tilde{m}_h \cdot e_3)(f_1) \|_{L^2(\omega)} + \| (\tilde{m}_h \cdot e_3)(f_2) \|_{L^2(\omega)} \right). \tag{4.31}
\]

From this it follows that

\[
\liminf \frac{1}{h^2} I_h \geq O(1) \limsup \left( \| (\tilde{m}_h \cdot e_3)(f_1) \|_{L^2(\omega)} + \| (\tilde{m}_h \cdot e_3)(f_2) \|_{L^2(\omega)} \right) = 0.
\]

The proposition is proved.

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