1 1D heat and wave equations on a finite interval

In this section we consider a general method of separation of variables and its applications to solving heat equation and wave equation on a finite interval (a_1, a_2). Since by translation we can always shift the problem to the interval $(0, a)$ we will be studying the problem on this interval.

1.1 Boundary conditions

The most common types of boundary conditions are

- **Dirichlet:** $u(0,t) = h(t), \quad u(a,t) = g(t)$.
- **Neumann:** $u_x(0,t) = h(t), \quad u_x(a,t) = g(t)$.
- **Mixed:** $u_x(0,t) = h(t), \quad u(a,t) = g(t)$ or $u(0,t) = h(t), \quad u_x(a,t) = g(t)$.
- **Periodic:** It is more convenient to consider the problem with periodic boundary conditions on the symmetric interval $(-a, a)$. Therefore boundary conditions in this case are
$$u(-a,t) = u(a,t), \quad u_x(-a,t) = u_x(a,t).$$

There is a generalization of mixed boundary condition sometimes called **Robin** boundary condition
$$\alpha u(0,t) + u_x(0,t) = h(t), \quad \beta u(a,t) + u_x(a,t) = g(t).$$
We will not be considering it here but the methods used below work for it as well.

1.2 Heat equation

Our goal is to solve the following problem

$$u_t = Du_{xx} + f(x,t), \quad x \in (0,a), \tag{1}$$

$$u(x,0) = \phi(x), \tag{2}$$

and u satisfies one of the above boundary conditions.

In order to achieve this goal we first consider a problem when $f(x,t) = 0$, $h(t) = 0$, $g(t) = 0$ and use the method of separation of variables to obtain solution.

To illustrate the method we solve the heat equation with Dirichlet and Neumann boundary conditions. Mixed and Periodic boundary conditions are treated in the similar way and we will use them in the section for wave equation.

1.2.1 Homogeneous heat equation with Dirichlet boundary conditions

Here we are solving the following problem

$$u_t = Du_{xx}, \quad \text{for } x \in (0,a), t > 0 \tag{3}$$

$$u(x,0) = \phi(x), \tag{4}$$

$$u(0,t) = 0, \quad u(a,t) = 0, \text{ for } t > 0. \tag{5}$$

The idea behind the method of separation of variables is to look for a solution in the following form

$$u(x,t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ will be determined. Let's assume that we have a solution $u(x,t)$ in the above form. Plugging it into the equation we obtain

$$X(x)T'(t) = DX''(x)T(t),$$

Dividing both parts by $DT(t)X(x)$ we arrive to the following equality

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)}.$$

Since the left side is a function of t only and the right side is the function of x only the above equality is possible if and only if

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)} = k$$

for some constant k. Therefore if $u(x,t) = X(x)T(t)$ is a solution of the heat equation then

$$X''(x) = kX(x), \quad \text{and} \quad T'(t) = kDT(t).$$

Moreover, from the boundary conditions (9) we know that $X(0)T(t) = 0$ and $X(a)T(t) = 0$ for all $t > 0$. This implies $X(0) = 0$ and $X(a) = 0$.

We can easily solve the problem for $T(t)$ to obtain

$$T(t) = Ce^{-kDt},$$

where we don't know constants C and k. The problem for $X(x)$ is called an eigenvalue problem

$$X''(x) = kX(x) \text{ on } (0,a), \quad X(0) = 0, \ X(a) = 0. \tag{6}$$

In order to solve it we have to find all k-s and all nontrivial solutions $X(x)$, corresponding to these k-s. It is not difficult to see that $k < 0$. Indeed, if we multiply both parts of the equation (34) by $X(x)$ and integrate by parts we obtain

$$\int_0^a |X'(x)|^2 + k|X(x)|^2 \, dx = 0.$$

Obviously, if $k \geq 0$ then $X(x) = 0$ on $(0,a)$. Since we are interested in non-trivial solutions we have to consider only $k = -\lambda^2 < 0$. In that case the general solution of (34) is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

and constants A, B, λ will be found from the boundary conditions $X(0) = X(a) = 0$. Let's consider $X(0) = 0$: obviously $X(0) = A$ and therefore $A = 0$. Taking this into account and using $X(a) = 0$ we obtain

$$B \sin(\lambda a) = 0.$$

It's clear that $B \neq 0$ as then the solution is zero everywhere and we are interested only in non-zer solutions. Therefore $\lambda = \frac{\pi n}{a}$ for all $n \in \mathbb{Z} \setminus \{0\}$. The solution of the eigenvalue problem is

$$k = \frac{\pi^2 n^2}{a^2},\ n \in \mathbb{N}^+ = \{1, 2, 3, ...\}, \quad X(x) = B \sin\left(\frac{\pi n}{a} x\right).$$

Combining our information about k, $X(x)$ and $T(t)$ we arrive to the conclusion that we have infinitely many solutions of the heat equation (7), satisfying the boundary conditions (9). Indeed, for all $n \in \mathbb{N}^+$ a function

$$u_n(x, t) = e^{-\frac{\pi^2 n^2}{a^2} Dt} \sin\left(\frac{\pi n}{a} x\right)$$

solves (7) and (9). Therefore, any linear combination of u_n-s also solves (7) and (9) and we represent

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2 n^2}{a^2} Dt} \sin\left(\frac{\pi n}{a} x\right),$$

where a_n are arbitrary constants. We will find a_n-s using initial data.

We can represent $\phi(x)$ in terms of a sine Fourier series

$$\phi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{a} x\right),$$

where

$$b_n = \frac{2}{a} \int_0^a \phi(x) \sin\left(\frac{\pi n}{a} x\right) dx.$$

Since we know that $u(x, 0) = \phi(x)$ we obtain

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n}{a} x\right) = \phi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{a} x\right).$$

Therefore

$$\sum_{n=1}^{\infty} (a_n - b_n) \sin\left(\frac{\pi n}{a} x\right) = 0.$$

Multiplying both parts of the above equality by $\sin\left(\frac{\pi m}{a} x\right)$, $m \in \mathbb{N}^+$ and integrating over $(0, a)$ we have $a_m = b_m$ for all $m \in \mathbb{N}^+$. Therefore we obtain that

$$a_n = \frac{2}{a} \int_0^a \phi(x) \sin\left(\frac{\pi n}{a} x\right) dx.$$

Finally, the solution to the problem (7), (36), (9) is

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2 n^2}{a^2} Dt} \sin\left(\frac{\pi n}{a} x\right),$$

where

$$a_n = \frac{2}{a} \int_0^a \phi(x) \sin\left(\frac{\pi n}{a} x\right) dx.$$

1.2.2 Homogeneous heat equation with Neumann boundary conditions

Here we are solving the following problem

$$u_t = Du_{xx}, \quad \text{for } x \in (0,a), t > 0 \tag{7}$$

$$u(x,0) = \phi(x), \tag{8}$$

$$u_x(0,t) = 0, \quad u_x(a,t) = 0, \text{ for } t > 0. \tag{9}$$

We use the same method os separation of variables and obtain the following eigenvalue problem for $X(x)$

$$X''(x) = kX(x), \text{ on } (0,a), \quad X'(0) = 0, \; X'(a) = 0. \tag{10}$$

In this case we can show by the same methods as before that $k = -\lambda^2 \leq 0$ and therefore the general solution is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

Using $X'(0) = 0$ we obtain $B = 0$ and using $X'(a) = 0$ we obtain $A\sin(\lambda a) = 0$. Therefore $\lambda = \frac{\pi n}{a}$ for all $n \in \mathbb{Z}$ and the solution of eigenvalue problem (10) is

$$k = \frac{\pi^2 n^2}{a^2}, \; n \in \mathbb{N} = \{0,1,2,3,...\}, \quad X(x) = A\cos\left(\frac{\pi n}{a}x\right).$$

Note that here we allow for $n = 0$ in which case $X(x) = A$ is a solution. As before we obtain the solution

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\frac{\pi^2 n^2}{a^2}Dt} \cos\left(\frac{\pi n}{a}x\right),$$

where we need to find a_n-s from initial data.

We can represent $\phi(x)$ in terms of a cosine Fourier series

$$\phi(x) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{\pi n}{a}x\right),$$

where

$$b_n = \frac{2}{a}\int_0^a \phi(x)\cos\left(\frac{\pi n}{a}x\right)dx \quad \text{for } n > 0,$$

$$b_0 = \frac{1}{a}\int_0^a \phi(x)\,dx.$$

As before using initial data we know that $a_n = b_n$ and therefore the solution is

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\frac{\pi^2 n^2}{a^2}Dt} \cos\left(\frac{\pi n}{a}x\right),$$

where

$$a_n = \frac{2}{a}\int_0^a \phi(x)\cos\left(\frac{\pi n}{a}x\right)dx \quad \text{for } n > 0,$$

$$a_0 = \frac{1}{a}\int_0^a \phi(x)\,dx.$$

1.2.3 Heat equation with $f(x,t) \neq 0$ and zero Dirichlet boundary conditions

Now let us explain how we can solve the following problem

$$u_t = Du_{xx} + f(x,t), \quad \text{for } x \in (0,a), t > 0 \tag{11}$$

$$u(x,0) = \phi(x), \tag{12}$$

$$u(0,t) = 0, \quad u(a,t) = 0, \text{ for } t > 0. \tag{13}$$

We know that a Fourier series with basis functions $\sin\left(\frac{\pi n}{a}x\right)$, $n \in \mathbb{N}^+$ is consistent with the boundary conditions. We also know that for a fixed t any function $u(x,t)$ and $f(x,t)$ can be expanded in a Fourier series as follows

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{\pi n}{a}x\right),$$

where

$$f_n(t) = \frac{2}{a}\int_0^a f(x,t)\sin\left(\frac{\pi n}{a}x\right)\,dx \quad \text{for } n \in \mathbb{N}^+,$$

and

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{\pi n}{a}x\right),$$

where $u_n(t)$ will be determined later using equation.

Plugging these representations for $u(x,t)$ and $f(x,t)$ in the (11) we obtain

$$\sum_{n=1}^{\infty}\left(u_n'(t) + D\frac{\pi^2 n^2}{a^2}u_n(t) - f_n(t)\right)\sin\left(\frac{\pi n}{a}x\right) = 0.$$

Multiplying both parts of equality by $\sin\left(\frac{\pi m}{a}x\right)$ and integrating over $(0,a)$ we have

$$u_m'(t) + D\frac{\pi^2 m^2}{a^2}u_m(t) = f_m(t).$$

Using initial data we clearly have

$$u(x,0) = \sum_{n=1}^{\infty} u_n(0)\sin\left(\frac{\pi n}{a}x\right) = \phi(x),$$

and therefore

$$u_m(0) = \frac{2}{a}\int_0^a \phi(x)\sin\left(\frac{\pi m}{a}x\right)\,dx \quad \text{for } m \in \mathbb{N}^+.$$

Now for each $m \in \mathbb{N}^+$ we have the following ODE

$$u_m'(t) + D\frac{\pi^2 m^2}{a^2}u_m(t) = f_m(t), \quad u_m(0) = \frac{2}{a}\int_0^a \phi(x)\sin\left(\frac{\pi m}{a}x\right)\,dx.$$

It is clear that we can solve it to obtain

$$u_m(t) = u_m(0)e^{-D\frac{\pi^2 m^2}{a^2}t} + e^{-D\frac{\pi^2 m^2}{a^2}t}\int_0^t e^{D\frac{\pi^2 m^2}{a^2}s} f_m(s)\,ds.$$

Since we know $u_m(0)$ and $f_m(t)$ we obtain $u_m(t)$ and therefore we have a solution of the problem (11)-(13) as

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{\pi n}{a}x\right).$$

We can use the same method to treat all other boundary conditions. I omit the details here.

1.2.4 Inhomogeneous boundary conditions

In this subsection we want to understand how to solve the following problem

$$u_t = Du_{xx}, \quad \text{for } x \in (0, a), t > 0 \tag{14}$$

$$u(x, 0) = 0, \tag{15}$$

$$u(0, t) = h(t), \quad u(a, t) = g(t), \text{ for } t > 0. \tag{16}$$

We can construct an auxiliary function

$$v(x, t) = \frac{g(t) - h(t)}{a} x + h(t).$$

It is clear that $w(x, t) = u(x, t) - v(x, t)$ satisfies the following problem

$$w_t = Dw_{xx} - v_t(x, t), \quad \text{for } x \in (0, a), t > 0 \tag{17}$$

$$w(x, 0) = -v(x, 0), \tag{18}$$

$$w(0, t) = 0, \quad w(a, t) = 0, \text{ for } t > 0. \tag{19}$$

Using results from the previous subsection we can find $w(x, t)$ and therefore we can find that

$$u(x, t) = w(x, t) + v(x, t),$$

where we know w and v.

Note that the same idea works for Neumann and Mixed boundary conditions. We just need to construct an appropriate auxiliary function $v(x, t)$ satisfying boundary conditions.

1.2.5 General problem

Now we are ready to solve the following general problem

$$u_t = Du_{xx} + f(x, t), \quad \text{for } x \in (0, a), t > 0 \tag{20}$$

$$u(x, 0) = \phi(x), \tag{21}$$

$$u(0, t) = h(t), \quad u(a, t) = g(t), \text{ for } t > 0. \tag{22}$$

It is straightforward to check that if $v(x, t)$ satisfies

$$v_t = Dv_{xx} + f(x, t), \quad \text{for } x \in (0, a), t > 0 \tag{23}$$

$$v(x, 0) = \phi(x), \tag{24}$$

$$v(0, t) = 0, \quad v(a, t) = 0, \text{ for } t > 0, \tag{25}$$

and $w(x, t)$ satisfies

$$w_t = Dw_{xx}, \quad \text{for } x \in (0, a), t > 0 \tag{26}$$

$$w(x, 0) = 0, \tag{27}$$

$$w(0, t) = h(t), \quad w(a, t) = g(t), \text{ for } t > 0, \tag{28}$$

then $u(x, t) = v(x, t) + w(x, t)$ solves (20), (21), (22). Since we already know from the previous subsections how to solve the problems for $v(x, t)$ and $w(x, t)$ we are done.

The same method works for all boundary conditions, not just Dirichlet.

1.3 Wave equation

Our goal is to solve the following problem

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad x \in (0,a), \tag{29}$$

$$u(x,0) = \phi(x), \; u_t(x,0) = \psi(x) \tag{30}$$

and u satisfies one of the boundary conditions.

In order to achieve this goal we proceed as with heat equation, first consider a problem when $f(x,t) = 0$, $h(t) = 0$, $g(t) = 0$ and use the method of separation of variables to obtain solution.

To illustrate the method we solve the wave equation with Mixed and Periodic boundary conditions. Dirichlet and Neumann are treated in the similar way and have been considered before for heat equation.

1.3.1 Homogeneous wave equation with mixed boundary conditions

Here we are solving the following problem

$$u_{tt} = c^2 u_{xx}, \quad \text{for } x \in (0,a), t > 0 \tag{31}$$

$$u(x,0) = \phi(x), \; u_t(x,0) = \psi(x) \tag{32}$$

$$u_x(0,t) = 0, \quad u(a,t) = 0, \; \text{for } t > 0. \tag{33}$$

We look for a solution in the following form

$$u(x,t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ will be determined. Let's assume that we have a solution $u(x,t)$ in the above form. Plugging it into the equation we obtain

$$X(x)T''(t) = c^2 X''(x)T(t),$$

Dividing both parts by $c^2 T(t)X(x)$ we arrive to the following equality

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left side is a function of t only and the right side is the function of x only the above equality is possible if and only if

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = k$$

for some constant k. Therefore if $u(x,t) = X(x)T(t)$ is a solution of the wave equation then

$$X''(x) = kX(x), \quad \text{and} \quad T''(t) = kc^2 T(t).$$

Moreover, from the boundary conditions (9) we know that $X'(0)T(t) = 0$ and $X(a)T(t) = 0$ for all $t > 0$. This implies $X'(0) = 0$ and $X(a) = 0$.

The problem for $X(x)$ is called an eigenvalue problem

$$X''(x) = kX(x) \text{ on } (0,a), \quad X'(0) = 0,\ X(a) = 0. \tag{34}$$

In order to solve it we have to find all k-s and all nontrivial solutions $X(x)$, corresponding to these k-s. It is not difficult to see that $k < 0$. Indeed, if we multiply both parts of the equation (34) by $X(x)$ and integrate by parts we obtain

$$\int_0^a |X'(x)|^2 + k|X(x)|^2\, dx = 0.$$

Obviously, if $k \geq 0$ then $X(x) = 0$ on $(0,a)$. Since we are interested in non-trivial solutions we have to consider only $k = -\lambda^2 < 0$. In that case the general solution of (34) is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

and constants A, B, λ will be found from the boundary conditions $X'(0) = X(a) = 0$. Let's consider $X'(0) = 0$: obviously $X'(0) = B\lambda$ and therefore $B = 0$. Taking this into account and using $X(a) = 0$ we obtain

$$A\cos(\lambda a) = 0.$$

It's clear that $A \neq 0$ as then the solution is zero everywhere and we are interested only in non-zero solutions. Therefore $\lambda = \frac{1}{a}\left(\frac{\pi}{2} + \pi n\right)$ for all $n \in \mathbb{Z}$. The solution of the eigenvalue problem is

$$k = \frac{1}{a^2}\left(\frac{\pi}{2} + \pi n\right)^2,\ n \in \mathbb{N} = \{0,1,2,3,...\}, \quad X(x) = A\cos\left(\frac{1}{a}\left(\frac{\pi}{2} + \pi n\right)x\right).$$

Solving equation for $T(t)$ we obtain

$$T(t) = C\cos\left(\frac{c}{a}\left(\frac{\pi}{2} + \pi n\right)t\right) + D\sin\left(\frac{c}{a}\left(\frac{\pi}{2} + \pi n\right)t\right).$$

Combining our information about $X(x)$ and $T(t)$ we arrive to the conclusion that we have infinitely many solutions of the wave equation (35), satisfying the boundary conditions (33). Indeed, for all $n \in \mathbb{N}$ a function

$$u_n(x,t) = \left(a_n\cos\left(\frac{c}{a}\left(\frac{\pi}{2} + \pi n\right)t\right) + b_n\sin\left(\frac{c}{a}\left(\frac{\pi}{2} + \pi n\right)t\right)\right)\cos\left(\frac{1}{a}\left(\frac{\pi}{2} + \pi n\right)x\right)$$

solves (35) and (33). Therefore, any linear combination of u_n-s also solves (35) and (33) and we represent

$$u(x,t) = \sum_{n=0}^{\infty}\left(a_n\cos\left(\frac{c}{a}\left(\frac{\pi}{2} + \pi n\right)t\right) + b_n\sin\left(\frac{c}{a}\left(\frac{\pi}{2} + \pi n\right)t\right)\right)\cos\left(\frac{1}{a}\left(\frac{\pi}{2} + \pi n\right)x\right),$$

where a_n, b_n are arbitrary constants. We will find a_n, b_n as before using initial data.

1.3.2 Homogeneous wave equation with periodic boundary conditions

Here we are solving the following problem

$$u_{tt} = c^2 u_{xx}, \quad \text{for } x \in (-a,a), t > 0 \tag{35}$$

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) \tag{36}$$

$$u(-a,t) = u(a,t), \quad u_x(-a,t) = u_x(a,t), \text{ for } t > 0. \tag{37}$$

We look for a solution in the following form

$$u(x,t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ will be determined. Let's assume that we have a solution $u(x,t)$ in the above form. Plugging it into the equation we obtain

$$X(x)T''(t) = c^2 X''(x)T(t),$$

Dividing both parts by $c^2T(t)X(x)$ we arrive to the following equality

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left side is a function of t only and the right side is the function of x only the above equality is possible if and only if

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = k$$

for some constant k. Therefore if $u(x,t) = X(x)T(t)$ is a solution of the wave equation then

$$X''(x) = kX(x), \quad \text{and} \quad T''(t) = kc^2T(t).$$

Moreover, from the boundary conditions (9) we know that $X(-a)T(t) = X(a)T(t)$ and $X'(-a)T(t) = X'(a)T(r)$ for all $t > 0$. This implies $X(-a) = X(a)$ and $X'(-a) = X'(a)$. Solving equation for $X(x)$ we obtain

$$X(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x).$$

Using boundary conditions we have

$$A\cos(\sqrt{k}a) - B\sin(\sqrt{(k)}a) = A\cos(\sqrt{k}a) + B\sin(\sqrt{k}a);$$

$$\sqrt{k}(A\sin(\sqrt{k}a) + B\cos(\sqrt{k}a)) = \sqrt{k}(-A\sin(\sqrt{k}a) + B\cos(\sqrt{k}a)).$$

Solving these equations we obtain

$$\sin(\sqrt{k}a) = 0.$$

Therefore $\sqrt{k} = \frac{\pi n}{a}$ for all $n \in \mathbb{N} = \{0, 1, 2, ...\}$ and

$$X_n(x) = A_n \cos\left(\frac{\pi n}{a}x\right) + B_n \sin\left(\frac{\pi n}{a}x\right).$$

Solving equation for $T(t)$ we obtain

$$T_n(t) = C_n \cos\left(\frac{\pi nc}{a}t\right) + D_n \sin\left(\frac{\pi nc}{a}t\right).$$

Now we can find the general solution $u(x,t)$ to be

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{\pi nc}{a}t\right) + D_n \sin\left(\frac{\pi nc}{a}t\right)\right)\left(A_n \cos\left(\frac{\pi n}{a}x\right) + B_n \sin\left(\frac{\pi n}{a}x\right)\right).$$

Using initial data we can recover a_0, A_n, B_n, C_n, D_n as before.

1.3.3 General problem

In order to solve the general problem (29), (30) with inhomogeneous boundary conditions we follow exactly the same ideas as in sections 1.2.4 and 1.2.5. The details are omitted.

2 1D heat and wave equations on $\mathbb{R}$

In this section we consider two general methods of solving heat and wave equations on $\mathbb{R}$, namely, method of fundamental solutions (it is also sometimes called method of Green's functions). We also can construct a solution on the semi-infinite domain $\mathbb{R}_+$ based on the knowledge of a solution on $\mathbb{R}$.

2.1 Boundary conditions

When we are on the whole $\mathbb{R}$ we cannot prescribe boundary conditions. Usually we assume some information about behaviour of the solution at infinity, for instance, that $u_x(x,t)$ or/and $u(x,t)$ vanish as $x \to \pm\infty$. We will spell out these assumptions for particular problems.

For semi-infinite domain $\mathbb{R}_+$ we prescribe Dirichlet or Neumann boundary conditions only at one end $x = 0$ and treat infinity as described above.

2.2 Heat equation on $\mathbb{R}$. Method of fundamental solution.

Our goal is to solve the following problem

$$u_t = Du_{xx} + f(x,t), \quad x \in \mathbb{R}, t > 0, \tag{38}$$

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}. \tag{39}$$

We also want to make reasonable assumptions about behaviour of $f(x,t)$ and $\phi(x)$ at infinity:

$$\int_{\mathbb{R}} |\phi(x)|\, dx < \infty, \quad \int_{\mathbb{R}} |f(x,t)|\, dx < \infty \quad \text{for all } t > 0.$$

Firstly, we will try to solve homogeneous equation without any initial data

$$u_t = Du_{xx}, \quad x \in \mathbb{R}, t > 0$$

using *similarity solutions*. It is clear that rescaling of $\mathbb{R}$ by a constant will always give the same domain $\mathbb{R}$. Therefore if we change space variable $z = ax$ and change time variable $s = a^2 t$ (it's called **dilation**), we will obtain exactly the same domains for x and t. Moreover we see that if $u(x,t)$ is a solution of the heat equation then

$$v(z,s) = Au(ax, a^2 t)$$

is also a solution of the above heat equation for general parameters a, A. We choose $A = a^\alpha$, where we will determine α later. Now we have two different solutions $v(z,s)$ and $u(x,t)$ and we notice that

$$\frac{z^2}{s} = \frac{x^2}{t}, \quad \frac{v(z,s)}{s^{\alpha/2}} = \frac{u(x,t)}{t^{\alpha/2}}.$$

This suggests to us that there are solutions such that these quantities are **invariant** with respect to **dilation** and therefore we can define

$$\xi = \frac{x}{\sqrt{t}}, \quad f(\xi) = \frac{u(x,t)}{t^{\alpha/2}}.$$

Therefore we look for *similarity solution* in the form

$$u(x,t) = f\left(\frac{x}{\sqrt{t}}\right) t^{\alpha/2}. \tag{40}$$

Plugging it into the heat equation we obtain the following equation for f

$$\frac{\alpha}{2} f(\xi) - \frac{1}{2}\xi f'(\xi) - D f''(\xi) = 0. \tag{41}$$

Now we have a choice of α that will give use solution to the (41) and hence a solution to heat equation. We would like to look for a solution that is integrable and decays at infinity together with its x-derivatives, i.e.

$$\int_{\mathbb{R}} |u(x,t)|\, dx < \infty, \quad u_x(x,t) \to 0 \text{ for } x \to \pm\infty.$$

If we impose these assumptions then integrating heat equation over $\mathbb{R}$ we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)\, dx = D \int_{\mathbb{R}} u_{xx}(x,t)\, dx = D(u_x(\infty,t) - u_x(-\infty,t)) = 0.$$

Therefore

$$\int_{\mathbb{R}} u(x,t)\, dx = const \quad \text{for all } t > 0.$$

Without loss of generality (by rescaling u) we can choose the above constant to be 1. Plugging in our anzatz (40) for $u(x,t)$ we obtain

$$t^{\alpha/2} \int_{\mathbb{R}} f\left(\frac{x}{\sqrt{t}}\right) dx = 1,$$

or

$$t^{(\alpha+1)/2} \int_{R} f(\xi)\, d\xi = 1 \quad \text{for all } t > 0.$$

Therefore we have to take $\alpha = -1$.

The equation (41) becomes

$$-\frac{1}{2} f(\xi) - \frac{1}{2}\xi f'(\xi) - D f''(\xi) = 0$$

and we can solve it as follows

$$\frac{1}{2}(\xi f(\xi))' + D f''(\xi) = \left(D f'(\xi) + \frac{1}{2}\xi f(\xi)\right)' = 0.$$

Using behavior at infinity we obtain

$$Df'(\xi) + \frac{1}{2}\xi f(\xi) = 0$$

and

$$f(\xi) = Ae^{-\frac{\xi^2}{4D}}.$$

Now recalling that $\int_R f(\xi)\,d\xi = 1$ we have $A = \frac{1}{\sqrt{4\pi D}}$ and

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}. \tag{42}$$

The solution in (42) is called *the fundamental solution of heat equation*.

In what follows below we will use fundamental solution to construct solutions to heat equation. We will denote the fundamental solution as

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}$$

and note that it has the following properties

1. $\Phi_t = D\Phi_{xx}$ for all $x \in \mathbb{R}$, $t > 0$;
2. $\Phi(x,t) = \Phi(-x,t) > 0$, for all $x \in \mathbb{R}$, $t > 0$;
3. $\Phi(x,t)$ is a smooth (C^∞) function of (x,t) for $x \in R$, $t > 0$;
4. $\int_R \Phi(x,t)\,dx = 1$ for all $t > 0$.
5. As $t \to 0^+$, $\Phi(x,t) \to 0$ for all $x \neq 0$ and $\Phi(0,t) \to \infty$.

Looking at the properties 4 and 5 we note that as $t \to 0^+$, $\Phi(x,t) \to \delta(x)$, where $\delta(x)$ is a delta-function. Note that we don't specify the convergence here as δ-function is not a function in a usual sense. In fact the convergence will be in a sense of "measures" but we don't discuss it here.

2.2.1 Homogeneous heat equation

In this section we solve the following problem

$$u_t = Du_{xx}, \quad x \in \mathbb{R}, t > 0, \tag{43}$$

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}. \tag{44}$$

For simplicity, we assume that $\phi(x)$ is uniformly continuous and bounded function. The result is true for any integrable function $\phi(x)$.

We note that if $\Phi(x,t)$ is the fundamental solution of heat equation then for any fixed $y \in \mathbb{R}$ the function $\Phi(x-y,t)$ also solves heat equation (59). Moreover we can take a convolution of Φ and ϕ and obtain that

$$v(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\phi(y)\,dy$$

satisfies (59). Recalling that as $t \to 0$, $\Phi(x-y,t) \to \delta(x-y)$ we expect that

$$v(x,t) \to \int_{\mathbb{R}} \delta(x-y)\phi(y)\,dy \quad \text{as } t \to 0.$$

Recalling properties of δ-function we have $\int_{\mathbb{R}} \delta(x-y)\phi(y)\,dy = \phi(x)$ and therefore $v(x,0) = \phi(x)$. It seems that if we take

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\phi(y)\,dy,$$

then it will be a solution of the problem (59), (60). Let's verify it.

It is clear that $u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\phi(y)\,dy$ satisfies (59) as

$$u_t - Du_x x = \int_{\mathbb{R}} \left[\Phi_t(x-y,t) - D\Phi_{xx}(x-y,t)\right]\phi(y)\,dy = 0.$$

We just need to check that

$$u(x,t) \to \phi(x) \quad \text{as } t \to 0^+.$$

Using property 4 of Φ it is clear that

$$u(x,t) - \phi(x) = \int_{\mathbb{R}} \Phi(x-y,t)(\phi(y)-\phi(x))\,dy.$$

As $\phi(x)$ is a uniformly continuous function we know that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|\phi(x) - \phi(y)| < \epsilon$ for all $|x-y| < \delta$. We now split our integral as follows

$$\int_{\mathbb{R}} \Phi(x-y,t)(\phi(y)-\phi(x))\,dy = \int_{|y-x|<\delta} \Phi(x-y,t)(\phi(y)-\phi(x))\,dy + \int_{|y-x|\geq\delta} \Phi(x-y,t)(\phi(y)-\phi(x))\,dy. \quad (45)$$

It is clear that

$$|\int_{|y-x|<\delta} \Phi(x-y,t)(\phi(y)-\phi(x))\,dy| \leq \int_{|y-x|<\delta} \Phi(x-y,t)|\phi(y)-\phi(x)|\,dy \leq \epsilon \int_{|y-x|<\delta} \Phi(x-y,t)\,dy \leq \epsilon \int_{\mathbb{R}} \Phi(x-y,t)\,dy = \epsilon \quad (46)$$

For the second integral we notice that as $\phi(x)$ is bounded, i.e. $|\phi(x)| \leq C$ for all $x \in \mathbb{R}$ then

$$|\int_{|y-x|\geq\delta} \Phi(x-y,t)(\phi(y)-\phi(x))\,dy| \leq 2C \int_{|y-x|\geq\delta} \Phi(x-y,t)\,dy = 4C \int_{\delta}^{\infty} \Phi(z,t)\,dz.$$

Using explicit expression for Φ we can easily compute

$$\int_{\delta}^{\infty} \Phi(z,t)\,dz = \frac{1}{\sqrt{4\pi Dt}} \int_{\delta}^{\infty} e^{-\frac{z^2}{4Dt}}\,dz = \frac{1}{\sqrt{\pi}} \int_{\frac{\delta}{\sqrt{4Dt}}}^{\infty} e^{-s^2}\,ds \leq \frac{1}{2\sqrt{\pi}} e^{-\frac{\delta}{\sqrt{4Dt}}}.$$

Combining everything we obtain that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\int_{\mathbb{R}} \Phi(x-y,t)(\phi(y)-\phi(x))\,dy\right| \leq \epsilon + \frac{2C}{\sqrt{\pi}} e^{-\frac{\delta}{\sqrt{4Dt}}}.$$

Taking limit as $t \to 0$ we have

$$\lim_{t\to 0^+} \left| \int_{\mathbb{R}} \Phi(x-y,t)(\phi(y)-\phi(x))\,dy \right| \le \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we take $\epsilon \to 0^+$ to obtain

$$\lim_{t\to 0^+} \int_{\mathbb{R}} \Phi(x-y,t)\phi(y)\,dy = \phi(x).$$

We just showed that $u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\phi(y)\,dy$ solves (59), (60).

2.2.2 Heat equation with $f(x,t) \neq 0$ and zero initial data

In this section we solve the following problem

$$u_t = Du_{xx} + f(x,t), \quad x \in \mathbb{R}, t > 0, \tag{47}$$

$$u(x,0) = 0, \quad x \in \mathbb{R}. \tag{48}$$

We will use *Duhamel's principle* to find the solution of the above problem. Let us start with the following auxiliary problem. Fix any $s > 0$ and consider

$$v_t = Dv_{xx}, \quad x \in \mathbb{R}, t > s, \tag{49}$$

$$v(x,s) = f(x,s), \quad x \in \mathbb{R}. \tag{50}$$

Due to translation invariance of the heat equation we can easily check that for each fixed s

$$v(x,t;s) = \int_{\mathbb{R}} \Phi(x-y,t-s)f(y,s)\,dy$$

solves the problem (64), (65) (note that $v(x,s;s) = f(x,s)$). Now we define the following function

$$w(x,t) = \int_0^t v(x,t;s)\,ds$$

and claim that it solves (47), (48). Let's verify it.

It is clear that $w(x,0) = 0$ and therefore (48) is satisfied. Now

$$w_t(x,t) = v(x,t;t) + \int_0^t v_t(x,t;s)\,ds = f(x,t) + \int_0^t v_t(x,t;s)\,ds$$

and

$$w_{xx}(x,t) = \int_0^t v_{xx}(x,t;s)\,ds.$$

Therefore for all $t > 0$

$$w_t - Dw_{xx} = f(x,t) + \int_0^t (v_t(x,t;s) - Dv_{xx}(x,t;s))\,ds = f(x,t).$$

We just showed that

$$u(x,t) = \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)f(y,s)\,dy\,ds$$

solves (47), (48).

2.2.3 General problem

Now we are ready to solve the following general problem

$$u_t = Du_{xx} + f(x,t), \quad \text{for } x \in \mathbb{R}, t > 0 \tag{51}$$

$$u(x,0) = \phi(x), \text{ for } x \in \mathbb{R}. \tag{52}$$

It is straightforward to check that if $v(x,t)$ satisfies

$$v_t = Dv_{xx}, \quad \text{for } x \in \mathbb{R}, t > 0 \tag{53}$$

$$v(x,0) = \phi(x), \tag{54}$$

and $w(x,t)$ satisfies

$$w_t = Dw_{xx} + f(x,t), \quad \text{for } x \in \mathbb{R}, t > 0 \tag{55}$$

$$w(x,0) = 0, \tag{56}$$

then $u(x,t) = v(x,t) + w(x,t)$ solves (51), (52). Since we already know from the previous subsections how to solve the problems for $v(x,t)$ and $w(x,t)$ we obtain

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y)\phi(y)\,dy + \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s) f(y,s)\,dy\,ds.$$

2.3 Wave equation on $\mathbb{R}$. Method of fundamental solution.

Our goal is to solve the following problem

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad x \in \mathbb{R}, t > 0 \tag{57}$$

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), \quad x \in \mathbb{R}. \tag{58}$$

We start by solving homogeneous problem with $f(x,t) \equiv 0$.

2.3.1 Homogeneous wave equation. D'Alembert's solution.

In this section we solve the following problem

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, t > 0, \tag{59}$$

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), \quad x \in \mathbb{R}. \tag{60}$$

Using the following change of variables $\xi = x - ct$, $\eta = x + ct$ we obtain

$$u_t = -cu_\xi + cu_\eta, \quad u_x = u_\xi + u_\eta,$$

$$u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}, \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Therefore

$$u_{tt} - c^2 u_{xx} = -2c^2 u_{\xi\eta} = 0.$$

It follows that $u = f(\xi) + g(\eta)$ and therefore returning to original variables

$$u(x,t) = f(x-ct) + g(x+ct).$$

Using initial data we obtain

$$f(x) + g(x) = \phi(x), \quad -cf'(x) + cg'(x) = \psi(x), \quad \text{for all } x \in \mathbb{R}.$$

After straightforward calculation it follows that for all $x \in \mathbb{R}$ and some fixed $a \in \mathbb{R}$

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_a^x \psi(s)\,ds, \quad g(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_a^x \psi(s)\,ds.$$

Now we see that

$$u(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_a^{x-ct} \psi(s)\,ds + \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_a^{x+ct} \psi(s)\,ds.$$

Finally we obtain D'Alembert's solution

$$u(x,t) = \frac{1}{2}\phi(x-ct) + \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds. \tag{61}$$

2.3.2 Wave equation with $f(x,t) \neq 0$ and zero initial data

In this section we solve the following problem

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad x \in \mathbb{R}, t > 0, \tag{62}$$

$$u(x,0) = 0, \ u_t(x,0) = 0 \quad x \in \mathbb{R}. \tag{63}$$

We again use *Duhamel's principle* to find the solution of the above problem. Let us start with the following auxiliary problem. Fix any $s > 0$ and consider

$$v_{tt} = c^2 v_{xx}, \quad x \in \mathbb{R}, t > s, \tag{64}$$

$$v(x,s) = 0, \ v_t(x,s) = f(x,s), \quad x \in \mathbb{R}. \tag{65}$$

Due to translation invariance of the wave equation we can easily check that for each fixed s

$$v(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(r,s)\,dr.$$

solves the problem (64), (65) (note that $v(x,s;s) = 0$, $v_t(x,s;s) = f(x,s)$). Now we define the following function

$$w(x,t) = \int_0^t v(x,t;s)\,ds$$

and claim that it solves (62), (63). Let's verify it.

It is clear that $w(x,0) = 0$ and therefore (63) is satisfied. Now

$$w_t(x,t) = v(x,t;t) + \int_0^t v_t(x,t;s)\,ds = \int_0^t v_t(x,t;s)\,ds,$$

$$w_{tt}(x,t) = v_t(x,t;t) + \int_0^t v_{tt}(x,t;s)\,ds = f(x,t) + \int_0^t v_{tt}(x,t;s)\,ds.$$

and

$$w_{xx}(x,t) = \int_0^t v_{xx}(x,t;s)\,ds.$$

Therefore for all $t > 0$

$$w_{tt} - c^2 w_{xx} = f(x,t) + \int_0^t (v_{tt}(x,t;s) - c^2 v_{xx}(x,t;s))\, ds = f(x,t).$$

We just showed that

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(r,s)\, dr\, ds$$

solves (62), (63).

2.3.3 General problem

Now we are ready to solve the following general problem

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad \text{for } x \in \mathbb{R}, t > 0 \tag{66}$$

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) \ \text{ for } x \in \mathbb{R}. \tag{67}$$

It is straightforward to check that if $v(x,t)$ satisfies

$$v_t = D v_{xx}, \quad \text{for } x \in \mathbb{R}, t > 0 \tag{68}$$

$$v(x,0) = \phi(x),\ v_t(x,0) = \psi(x) \ \text{ for } x \in \mathbb{R}. \tag{69}$$

and $w(x,t)$ satisfies

$$w_t = D w_{xx} + f(x,t), \quad \text{for } x \in \mathbb{R}, t > 0 \tag{70}$$

$$w(x,0) = 0, w_t(x,0) = 0 \ \text{ for } x \in \mathbb{R} \tag{71}$$

then $u(x,t) = v(x,t) + w(x,t)$ solves (66), (67). Since we already know from the previous subsections how to solve the problems for $v(x,t)$ and $w(x,t)$ we obtain

$$u(x,t) = \frac{1}{2}\phi(x-ct) + \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(r)\, dr + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(r,s)\, dr\, ds.$$