The Normal density is normed

Márton Balázs* and Bálint Tóth*

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This little write-up is part of important foundations of probability that were left out of the unit Probability 1 due to lack of time and prerequisites. Here we show that the Standard Normal density function is indeed a density. The Probability 1 unit skips this as it requires two dimensional calculus and polar coordinates. We assume a working knowledge of these.

Recall the form

$$\frac{1}{\sqrt{2\pi}\cdot\sigma}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x\in\mathbb{R})$$

of the $\mathcal{N}(\mu,\,\sigma^2)$ density. It is clearly non-negative, to verify that it is indeed a density, we prove

Proposition 1 *Let $\mu\in\mathbb{R}$, $\sigma>0$ be real parameters, then*

$$\frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x=1.$$

Proof. Let I be the integral in question. With the substitution $y=[x-\mu]/\sigma$

$$I=\frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y,$$

the integral of the Standard Normal density. The antiderivative cannot be written in an explicit form. However, we can apply the trick below to compute the integral over the whole real line.

$$I^2=\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y\Big)\cdot\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{z^2}{2}}\,\mathrm{d}z\Big)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{y^2+z^2}{2}}\,\mathrm{d}y\,\mathrm{d}z.$$

Introducing the polar coordinates: $y=r\cos\theta,\ z=r\sin\theta,\ \mathrm{d}y\,\mathrm{d}z=r\,\mathrm{d}r\,\mathrm{d}\theta$,

$$I^2=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\mathrm{e}^{-\frac{r^2}{2}}\,r\,\mathrm{d}r\,\mathrm{d}\theta=\Big[-\mathrm{e}^{-\frac{r^2}{2}}\Big]_0^{\infty}=1.$$

As $I>0$, this completes the proof. □

*University of Bristol / Budapest University of Technology and Economics