Probability 1, Autumn 2016, Assessed HW sheet 1

Deadline: To be handed in by 10am on Friday the 4th November.
Hand in: You should hand your work along with a completed cover sheet in to the marked cabinet on the ground floor of the School of mathematics. Please staple each sheet of your work together with the cover page at the front. Cover sheets are available on the ground floor of the School of Mathematics.
Assessment: This homework will count for 5% of your total mark for Probability 1. To obtain full marks you will need to explain clearly how you obtained your answer, using appropriate notation.
Collaboration: The work you hand in should be your own work. You are welcome to discuss the problems with each other but the solutions you hand in should be written solely by you.
Solutions will be available on Blackboard on the 12th Nov.

HW A1.1 A biased coin that comes up Heads with probability p is flipped n times. Denote by A_i the event that the i^{th} flip results in Heads ($i = 1 \dots n$).

(a) (**2 marks**) Calculate the probability of $\bigcup_{i=1}^{n} A_i$ by applying De Morgan's Law and then independence of the A_i's.

(b) (**4 marks**) Calculate the probability of $\bigcup_{i=1}^{n} A_i$ by applying the inclusion-exclusion principle first and then independence of the A_i's.

(c) (**1 marks**) By comparing your answers in the previous parts, prove the identity

$$(1-p)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k p^k$$

for any $n > 0$ integer and $0 \le p \le 1$ real number.

(d) (**2 marks**) From the previous display, can you prove

$$(a-b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (-b)^k$$

for arbitrary positive reals a and b?

HW A1.2 (**8 marks**) In a probability lecture 20 students sit in the first row of seats and further 280 students sit in the seats behind. The instructor plays the birthday problem with them: he asks the students one by one in the first row about their birthdays (day, month; but not the year) and lets any student in the classroom interrupt if they hear their own birthdays said. Find the probability that no coinciding birthdays occur (within the first row and between the first row and rest of the classroom) by the time all 20 students in the first row are asked. Assume no one was born on the 29^{th} February. Also, unlike in class, the instructor is not part of the game, he does not reveal his own birthday.

HW A1.3 (**8 marks**) A coin comes up Heads with probability p and Tails with probability $1-p$. Flipping this coin four times the sequence of outcomes is noted and then rewritten by replacing Heads by 0's and Tails by 1's. This way, a sequence of length four that consists of 0's and 1's, is obtained. Let X be the number that has this sequence as its base 2 expansion. Example: if the flips are Heads, Tails, Tails, Heads, then the 0-1 sequence is 0110 and $X = 0 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1 = 6$. Plot the probability mass function of X and determine its expectation $\mathbf{E}X$ in case

(a) $p = \frac{1}{2}$, the coin is fair;

(b) $p = \frac{1}{3}$, the coin is biased.