Probability 1, Autumn 2016, Assessed HW sheet 2

Deadline: To be handed in by 10am on Friday the 2nd December.
Hand in: You should hand your work along with a completed cover sheet in to the marked cabinet on the ground floor of the School of mathematics. Please staple each sheet of your work together with the cover page at the front. Cover sheets are available on the ground floor of the School of Mathematics.
Assessment: This homework will count for 5% of your total mark for Probability 1. To obtain full marks you will need to explain clearly how you obtained your answer, using appropriate notation.
Collaboration: The work you hand in should be your own work. You are welcome to discuss the problems with each other but the solutions you hand in should be written solely by you.
Solutions will be available on Blackboard on the 10th December.

HW A2.1 (a) (**4 marks**) Rolling two fair dice, let E be the event that the sum of the two numbers shown is m, and F the event that the first die shows i. (Here $m = 2, 3, \ldots, 12$, $i = 1, 2, \ldots, 6$.) We have seen in lectures that these events are independent when $m = 7$ and $i = 3$. List all other choices of m and i which also make these events independent.

(b) (**4 marks**) Repeat the problem if the dice are four sided (and $m = 2, 3, \ldots, 8$, $i = 1, 2, 3, 4$).

HW A2.2 (a) (**1 marks**) The k^{th} *centered moment* of a random variable is defined, if exists, by $\mathbf{E}\big((X - \mathbf{E}X)^k\big)$. Identify the first and second centered moments of a random variable (assume they exist).

(b) (**2 marks**) Let $X \sim \text{Binom}(n, p)$. What is the distribution of $Y = n - X$?

(c) (**3 marks**) Let $X \sim \text{Binom}(n, p)$, and Y as above. Show that $\mathbf{Var}Y = \mathbf{Var}X$.

(d) (**2 marks**) Let $X \sim \text{Binom}(n, p)$. Use some of the above to show that the k^{th} centered moment of X, as a function of p, is invariant to a $p \leftrightarrow 1 - p$ swap when k is even, and switches sign under this swap when k is odd.

HW A2.3 (a) (**4 marks**) Let X be a standard normal variable, and a a non-random real number. Calculate $\mathbf{E}\big((X - a)^+\big)$, where $(X - a)^+$ is the *positive part* of $X - a$:

$$(X - a)^+ = \begin{cases} X - a, & \text{if } X \ge a; \\ 0, & \text{if } X \le a. \end{cases}$$

(b) (**3 marks**) A building contractor expects to complete a project on a certain *target date*. Due to uncertainties, they calculate with a delay X relative to the target date. X could be negative (completing the project earlier than planned), zero (completing on the target date), or positive (delivering the project later than expected), and they assume X, measured in weeks, is standard normal. The manager signs a contract about completing the project on a *contract date*, which is a weeks relative to the company's target date. This a can be any real number, e.g., negative (contract date is before target date), zero (contract date agrees with target date), or positive (contract date is after target date). Finishing too early has drawbacks, and the manager estimates it costs the company a weekly rate of £1000 to complete the project earlier than the contract date. Obviously, finishing later than contracted is even worse, with a cost of £2000 per week for delays after the contract date. (These costs change continuously in time.) Use part (a) to calculate the expected cost coming from the difference between completion and the contract date, as a function of a.

(c) (**2 marks**) Find the value of a that minimises the expected cost calculated in part (b). How much is the minimal expected cost?